

REMARKS ON THE METRIC INDUCED BY THE ROBIN FUNCTION II

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ABSTRACT. Let D be a smoothly bounded pseudoconvex domain in \mathbf{C}^n , $n > 1$. Using the Robin function $\Lambda(p)$ that arises from the Green function $G(z, p)$ for D with pole at $p \in D$ associated with the standard sum-of-squares Laplacian, N. Levenberg and H. Yamaguchi had constructed a Kähler metric (the so-called Λ -metric) on D . Assume that D is strongly pseudoconvex and ds^2 denotes the Λ -metric on D . In this article, first we prove that the holomorphic sectional curvature of ds^2 along normal directions converges to a negative constant near the boundary of D . Then, we prove that if D is not simply connected, then any nontrivial homotopy class of $\pi_1(D)$ contains a closed geodesic for ds^2 . Finally, we prove that the dimension of the space of square integrable harmonic (p, q) -forms on D relative to ds^2 is zero except when $p + q = n$ in which case it is infinite.

1. INTRODUCTION

Let D be a C^∞ -smoothly bounded domain in \mathbf{C}^n ($n \geq 2$). For $p \in D$, let $G(z, p)$ be the Green function for D with pole at p associated to the standard Laplacian

$$\Delta = 4 \sum_{i=1}^n \frac{\partial^2}{\partial z_i \partial \bar{z}_i}$$

on $\mathbf{C}^n \approx \mathbf{R}^{2n}$. Then $G(z, p)$ is the unique function of $z \in D$ satisfying $G(z, p)$ is harmonic on $D \setminus \{p\}$, $G(z, p) \rightarrow 0$ as $z \rightarrow \partial D$ and $G(z, p) - |z - p|^{-2n+2}$ is harmonic near p . Thus

$$\Lambda(p) = \lim_{z \rightarrow p} (G(z, p) - |z - p|^{-2n+2})$$

exists and is called the Robin constant for D at p . The function

$$\Lambda : p \rightarrow \Lambda(p)$$

is called the Robin function for D .

The Robin function for D is negative, real analytic and tends to $-\infty$ near ∂D (see [10]). Further, if D is pseudoconvex then by a result of Levenberg-Yamaguchi ([7]), $\log(-\Lambda)$ is a strongly plurisubharmonic function on D . Therefore

$$ds^2 = \sum_{\alpha, \beta=1}^n \frac{\partial^2 \log(-\Lambda)}{\partial z_\alpha \partial \bar{z}_\beta} dz_\alpha \otimes d\bar{z}_\beta$$

is a Kähler metric on D which is called the Λ -metric. Recall that the holomorphic sectional curvature of ds^2 at $z \in D$ along the direction $v \in \mathbf{C}^n$ is given by

$$R(z, v) = \frac{R_{\alpha\bar{\beta}\gamma\bar{\delta}} v^\alpha \bar{v}^\beta v^\gamma \bar{v}^\delta}{g_{\alpha\bar{\beta}} v^\alpha \bar{v}^\beta}$$

where

$$R_{\alpha\bar{\beta}\gamma\bar{\delta}} = -\frac{\partial^2 g_{\alpha\bar{\beta}}}{\partial z_\gamma \partial \bar{z}_\delta} + g^{\nu\bar{\mu}} \frac{\partial g_{\alpha\bar{\mu}}}{\partial z_\gamma} \frac{\partial g_{\nu\bar{\beta}}}{\partial \bar{z}_\delta}$$

are the components of the curvature tensor,

$$g_{\alpha\bar{\beta}} = \frac{\partial^2 \log(-\Lambda)}{\partial z_\alpha \partial \bar{z}_\beta}$$

are the components of ds^2 and $g^{\alpha\bar{\beta}}$ are the entries of the matrix $(g_{\alpha\bar{\beta}})^{-1}$. In the above formulae, the standard convention of summing over all indices that appear once in the upper and lower position is being followed.

Now, let v be a vector in \mathbf{C}^n . At each point $z \in \partial D$, there is a canonical splitting $\mathbf{C}^n = H_z(\partial D) \oplus N_z(\partial D)$ along the complex tangential and normal directions at z and hence v can uniquely be written as $v = v_H(z) + v_N(z)$ where $v_H(z) \in H_z(\partial D)$ and $v_N(z) \in N_z(\partial D)$. Also, the smoothness of ∂D implies

that if $z \in D$ is sufficiently close to ∂D , then there is a unique point $\pi(z) \in \partial D$ that is closest to it, i.e., $d(z, \partial D) = |z - \pi(z)|$. Therefore, v can uniquely be written as $v = v_H(\pi(z)) + v_N(\pi(z))$. We will abbreviate $v_H(\pi(z))$ as $v_H(z)$ and $v_N(\pi(z))$ as $v_N(z)$. For a strongly pseudoconvex domain D , the boundary behaviour of $R(z, v_N(z))$ was calculated in [1] in a special case, viz., when $z \rightarrow z_0 \in \partial D$ along the inner normal to ∂D at z_0 . The purpose of this article is threefold. One, we remove the restriction that $z \rightarrow z_0$ along the inner normal in obtaining the boundary behaviour of $R(z, v_N(z))$. More precisely, we have the following:

Theorem 1.1. *Let D be a C^∞ -smoothly bounded strongly pseudoconvex domain in \mathbf{C}^n . Fix $z_0 \in \partial D$ and let $v \in \mathbf{C}^n$. Then for $z \in D$*

$$\lim_{z \rightarrow z_0} R(z, v_N(z)) = -\frac{1}{n-1}.$$

To understand the difficulty in the computation, let us assume without loss of generality that $z_0 = 0$ and the normal to ∂D at z_0 is along the $\Re z_n$ -axis. Let $\{z_\nu\}$ be a sequence of points in D converging to 0. Without loss of generality, let us assume that the distance between z_ν and ∂D , denoted by δ_ν , is realised by a unique point $\pi(z_\nu) \in \partial D$, i.e.,

$$\delta_\nu = d(z_\nu, \partial D) = |z_\nu - \pi(z_\nu)|$$

for all $\nu \geq 1$. Now for each ν , apply a translation τ_ν to D followed by a unitary rotation σ_ν to obtain a new domain D_ν so that $\pi(z_\nu) \in \partial D$ corresponds to $0 \in \partial D_\nu$ and the normal to ∂D_ν at 0 is along the $\Re z_n$ axis. We will denote the composition $\sigma_\nu \circ \tau_\nu$ by θ_ν . Note that under the map θ_ν , $z_\nu \in D$ corresponds to $p_\nu = (0, \dots, -\delta_\nu) \in D_\nu$ and

$$\theta'_\nu(z_\nu)v_N(z_\nu) = (0, \dots, 0, |v_N(z_\nu)|).$$

Therefore, by the invariance of the Λ -metric under translation and unitary rotation [1, lemma 5.1],

$$(1.1) \quad R_D(z_\nu, v_N(z_\nu)) = R_{D_\nu}(p_\nu, (0, \dots, 0, |v_N(z_\nu)|)) \\ = \frac{1}{(g_{\nu n \bar{n}}(p_\nu))^2} \left(-\frac{\partial^2 g_{\nu n \bar{n}}}{\partial z_n \partial \bar{z}_n}(p_\nu) + \sum_{\alpha, \beta=1}^n g_{\nu}^{\beta \bar{\alpha}}(p_\nu) \frac{\partial g_{\nu n \bar{\alpha}}}{\partial z_n}(p_\nu) \frac{\partial g_{\nu \beta \bar{n}}}{\partial \bar{z}_n} \right)$$

where

$$(1.2) \quad g_{\nu \alpha \bar{\beta}} = \frac{\partial^2 \log(-\Lambda_\nu)}{\partial z_\alpha \partial \bar{z}_\beta}$$

are the components of the Λ -metric ds_ν^2 on D_ν and $g_{\nu}^{\alpha \bar{\beta}}$ are the entries of the matrix $(g_{\nu \alpha \bar{\beta}})^{-1}$. To compute the limit of the right hand side of (1.1) we have to find the asymptotics of the metric components $g_{\nu \alpha \bar{\beta}}$ and their derivatives along the sequence $\{p_\nu\}$. From (1.2), it is natural to hope that this can be achieved by computing the asymptotics of Λ_ν and their derivatives along $\{p_\nu\}$. To be more precise, let ψ be a C^∞ -smooth function on \mathbf{C}^n that defines the domain D and $\bar{\partial}\psi(0) = (0, \dots, 1)$. Then for each $\nu \geq 1$, $\psi_\nu = \psi \circ \theta_\nu^{-1}$ is a C^∞ -smooth defining function for D_ν . Also, it is evident that $\{\psi_\nu\}$ converges in the C^∞ -topology on compact subsets of \mathbf{C}^n to ψ . We then want to compute the rate of growth of

$$D^{A\bar{B}}\Lambda_\nu(p_\nu) = \frac{\partial^{|A|+|B|}\Lambda_\nu}{\partial z_1^{\alpha_1} \dots \partial z_n^{\alpha_n} \partial \bar{z}_1^{\beta_1} \dots \partial \bar{z}_n^{\beta_n}}(p_\nu), \quad A = (\alpha_1, \dots, \alpha_n), B = (\beta_1, \dots, \beta_n) \in \mathbf{N}^n$$

in terms of $\psi_\nu(p_\nu)$. In this regard, we prove the following:

Theorem 1.2. *Let D be a C^∞ -smoothly bounded domain in \mathbf{C}^n and let ψ be a C^∞ -smooth defining function for D defined on all of \mathbf{C}^n . Let $\{D_\nu\}$ be a sequence of C^∞ -smoothly bounded domains in \mathbf{C}^n with defining functions ψ_ν that converge in the C^∞ -topology on compact subsets of \mathbf{C}^n to ψ . Let $p_\nu \in D_\nu$ be such that $\{p_\nu\}$ converges to $p_0 \in \partial D$. Define the half space*

$$\mathcal{H} = \left\{ w \in \mathbf{C}^n : 2\Re \left(\sum_{\alpha=1}^n \psi_\alpha(p_0) w_\alpha \right) - 1 < 0 \right\}$$

and let $\Lambda_{\mathcal{H}}$ denotes the Robin function for \mathcal{H} . Then

$$(-1)^{|A|+|B|} D^{A\bar{B}}\Lambda_\nu(p_\nu) (\psi_\nu(p_\nu))^{2n-2+|A|+|B|} \rightarrow D^{A\bar{B}}\Lambda_{\mathcal{H}}(p_0)$$

as $\nu \rightarrow \infty$.

We will show in section 6 that the asymptotics obtained in the above theorem suffice to calculate the limit of the first term of (1.1). However it turns out that the second term remains indeterminate by these asymptotics. This means that in order to calculate this term we need finer asymptotics of Λ_ν and their derivatives. A similar situation was handled in [1] by using the following result of Levenberg-Yamaguchi [7]: The function λ defined by

$$(1.3) \quad \lambda(p) = \begin{cases} \Lambda(p)(\psi(p))^{2n-2} & ; \quad \text{if } p \in D \\ -|\partial\psi(p)|^{2n-2} & ; \quad \text{if } p \in \partial D \end{cases}$$

is C^2 up to \overline{D} . We will call λ the normalised Robin function associated to (D, ψ) . Thus it is expected that finer asymptotics of Λ_ν and their derivatives along $\{p_\nu\}$ could be obtained if the functions $\lambda_\nu = \Lambda_\nu \psi_\nu$ and their derivatives along $\{p_\nu\}$ are bounded. While $\lambda_\nu(p_\nu)$ converge to $\lambda(p_0)$ by theorem 1.2, we establish the convergence of first and second derivatives of λ_ν along $\{p_\nu\}$ in the following:

Theorem 1.3. *Under the hypothesis of theorem 1.2, we have*

$$(1) \quad \lim_{\nu \rightarrow \infty} \frac{\partial \lambda_\nu}{\partial p_\alpha}(p_\nu) = \frac{\partial \lambda}{\partial p_\alpha}(p_0), \text{ and}$$

$$(2) \quad \lim_{\nu \rightarrow \infty} \frac{\partial^2 \lambda_\nu}{\partial p_\alpha \partial \overline{p}_\beta}(p_\nu) = \frac{\partial^2 \lambda}{\partial p_\alpha \partial \overline{p}_\beta}(p_0).$$

where λ is the normalised Robin function associated to (D, ψ) and λ_ν is the normalised Robin function associated to (D_ν, ψ_ν) .

We remark that unlike the Bergman, Carathéodory and Kobayashi metrics, the Λ -metric is not invariant under biholomorphisms in general. For an example we refer to [1]. The only information in this respect that we have is that any biholomorphism between two C^∞ -smoothly bounded strongly pseudoconvex domains is Lipschitz with respect to the Λ -metric. This follows from [1, theorem 1.4]. Despite this drawback, we put our effort to explore this metric by finding its various properties analogous to those possessed by these invariant metrics. The second theme of this article is to study the existence of closed geodesics for the Λ -metric of a given homotopy type. In [6], Herbort proved that on a C^∞ -smoothly bounded strongly pseudoconvex domain D in \mathbf{C}^n which is not simply connected, every nontrivial homotopy class in $\pi_1(D)$ contains a closed geodesic for the Bergman metric. Using the asymptotics of the Λ -metric derived in [1] we prove the following analogue for the Λ -metric:

Theorem 1.4. *Let D be a C^∞ -smoothly bounded strongly pseudoconvex domain in \mathbf{C}^n which is not simply connected. Then every nontrivial homotopy class in $\pi_1(D)$ contains a closed geodesic for the Λ -metric.*

Let D be C^∞ -smoothly bounded strongly pseudoconvex domain in \mathbf{C}^n . H. Donnelly and C. Fefferman [4] proved that D does not admit any square integrable harmonic (p, q) -form relative to the Bergman metric except when $p + q = n$, in which case the space of such forms is infinite dimensional. A more transparent and elementary proof of the infinite dimensionality of the L^2 -cohomology of the middle dimension was given by Ohsawa [9]. In [3], Donnelly gave an alternative proof of the vanishing of the L^2 -cohomology outside the middle dimension using the following observation of Gromov [5]: If M is a complete Kähler manifold of complex dimension n such that the Kähler form ω of M can be written as $\omega = d\eta$, where η is bounded in supremum norm, then M does not admit any square integrable harmonic i form for $i \neq n$. Finally, we observe that these ideas can be applied to the Λ -metric to prove the following:

Theorem 1.5. *Let D be a C^∞ -smoothly bounded strongly pseudoconvex domain in \mathbf{C}^n . Let $\mathcal{H}_2^{p,q}(D)$ be the space of square integrable harmonic (p, q) -forms relative to the Λ -metric. Then*

$$\dim \mathcal{H}_2^{p,q}(D) = \begin{cases} 0 & ; \quad \text{if } p + q \neq n, \\ \infty & ; \quad \text{if } p + q = n. \end{cases}$$

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2. PROPERTIES OF λ

Let D be a C^∞ -smoothly bounded domain in \mathbf{C}^n with a C^∞ -smooth defining function ψ defined on all of \mathbf{C}^n . In this section, we recall some basic properties of the normalised Robin function λ associated to (D, ψ) . We start by describing the geometric meaning of $\lambda(p)$. Given $p \in D$, let

$$T : D \times \mathbf{C}^n \rightarrow \mathbf{C}^n$$

be the map defined by

$$(2.1) \quad T(p, z) = \frac{z - p}{-\psi(p)}.$$

Set

$$(2.2) \quad D(p) = \begin{cases} T(p, D) & ; \text{ if } p \in D, \\ \left\{ w \in \mathbf{C}^n : 2\Re(\sum_{\alpha=1}^n \psi_\alpha(p)w_\alpha) - 1 < 0 \right\} & ; \text{ if } p \in \partial D. \end{cases}$$

Thus $\{D(p) : p \in \overline{D}\}$ is a family of domains in \mathbf{C}^n each containing the origin. When $p \in D$, $D(p)$ is the image of D under the affine transformation $T(p, \cdot)$ and hence by [10, proposition 5.1], we have

$$\Lambda_{D(p)}(0) = \Lambda(p)(\psi(p))^{2n-2} = \lambda(p).$$

When $p \in \partial D$, $D(p)$ is a half space for which we have the explicit formula [1, (1.4)]

$$\Lambda_{D(p)}(0) = -|\partial\psi(p)|^{2n-2} = \lambda(p).$$

Thus for each $p \in \overline{D}$, $\lambda(p)$ is the Robin constant for $D(p)$ at the origin. We will denote the Green function for $D(p)$ with pole at p by $g(p, w)$.

To discuss the regularity of the function $\lambda(p)$ on \overline{D} , we set

$$\mathcal{D} = \cup_{p \in D} (p, D(p)) = \{(p, w) : p \in D, w \in D(p)\}.$$

The set \mathcal{D} can be considered as a variation of domains in \mathbf{C}^n with parameter space D , i.e., as a map

$$\mathcal{D} : p \rightarrow D(p)$$

which associates to each $p \in D$ a domain $D(p) \subset \mathbf{C}^n$. We call $\mathcal{D} : p \rightarrow D(p)$ the variation associated to (D, ψ) . The following function

$$(2.3) \quad f(p, w) = 2\Re \left\{ \sum_{\alpha=1}^n \int_0^1 \left(w_\alpha \psi_\alpha(p - \psi(p)tw) \right) dt \right\} - 1$$

was constructed in [7] which is jointly smooth on $\mathbf{C}^n \times \mathbf{C}^n$ and satisfies, taking $\tilde{\mathcal{D}} = D \times \mathbf{C}^n$,

- (i) $\mathcal{D} = \{(p, w) \in \tilde{\mathcal{D}} : f(p, w) < 0\}$, $\partial\mathcal{D} := \{(p, w) : p \in D, w \in \partial D(p)\} = \{(p, w) \in \tilde{\mathcal{D}} : f(p, w) = 0\}$ and $\text{Grad}_{(p,w)} f \neq 0$ on $\partial\mathcal{D}$,
- (ii) For each $p \in D$, $D(p) = \{w \in \mathbf{C}^n : f(p, w) < 0\}$, $\partial D(p) = \{w \in \mathbf{C}^n : f(p, w) = 0\}$ and $\text{Grad}_w f(p, w) \neq 0$ on $\partial D(p)$.

Therefore, we say that the variation $\mathcal{D} : p \rightarrow D(p)$ is smooth and is defined by $f(p, w)$. It is evident that the variation

$$\mathcal{D} \cup \partial\mathcal{D} : p \rightarrow D(p) \cup \partial D(p) = \overline{D}(p)$$

is diffeomorphically equivalent to the trivial variation $D \times \overline{D}$. It follows that $g(p, w)$ has a C^4 extension to a neighbourhood of $\mathcal{D} \setminus D \times \{0\}$. Now fix a point $p_0 \in D$ and let $\overline{B}(0, r) \subset D(p_0)$. Then there exists a neighbourhood U of p_0 in D such that $\overline{B}(0, r) \subset D(p)$ for all $p \in U$. Since $g(p, w) - |w|^{-2n+2}$ is a harmonic function of $w \in D(p)$ and is equal to $\lambda(p)$ when $w = 0$, we obtain by the mean value property of harmonic function

$$(2.4) \quad \begin{aligned} \lambda(p) &= \frac{1}{r^{2n-1}\sigma_{2n}} \int_{\partial B(0,r)} (g(p, w) - |w|^{-2n+2}) dS_w \\ &= -\frac{1}{r^{2n-2}} + \frac{1}{r^{2n-1}\sigma_{2n}} \int_{\partial B(0,r)} g(p, w) dS_w \end{aligned}$$

where by dS we denote the surface area measure on a smooth surface in \mathbf{R}^{2n} and σ_{2n} be the surface area of $\partial B(0, 1)$. It follows that $\lambda(p)$ is smooth on U and thus on D .

Now let $1 \leq \gamma \leq n$. Observe that for each $p \in D$, the functions

$$\frac{\partial g}{\partial p_\gamma}(p, w), \quad \frac{\partial^2 g}{\partial p_\gamma \partial \overline{p}_\gamma}(p, w)$$

are harmonic in all of $D(p)$ and

$$\frac{\partial g}{\partial p_\gamma}(p, 0) = \frac{\partial \lambda}{\partial p_\gamma}(p), \quad \frac{\partial^2 g}{\partial p_\gamma \partial \overline{p}_\beta}(p, 0) = \frac{\partial^2 \lambda}{\partial p_\gamma \partial \overline{p}_\beta}.$$

To find the boundary values of these functions in terms of f , consider the quantities k_1^γ and k_2^γ ,

$$(2.5) \quad k_1^\gamma(p, w) = \frac{\partial f}{\partial p_\gamma}(p, w) |\partial_w f(p, w)|^{-1} \quad \text{and} \quad k_2^\gamma(p, w) = \mathcal{L}^\gamma f(p, w) |\partial_w f(p, w)|^{-3}$$

where

$$(2.6) \quad \mathcal{L}^\gamma f = \frac{\partial^2 f}{\partial p_\gamma \partial \bar{p}_\gamma} |\partial_w f|^2 - 2\Re \left(\frac{\partial f}{\partial p_\gamma} \sum_{\alpha=1}^n \frac{\partial f}{\partial \bar{w}_\alpha} \frac{\partial^2 f}{\partial w_\alpha \partial \bar{p}_\gamma} \right) + \left| \frac{\partial f}{\partial p_\gamma} \right|^2 \Delta_w f,$$

defined wherever $\partial_w f(p, w) > 0$, thus, in particular on

$$\partial \mathcal{D} = \cup_{p \in D} (p, \partial D(p)).$$

Note that on $\partial \mathcal{D}$, the quantities k_1^γ and k_2^γ are independent of the defining function f for \mathcal{D} . Since $g(p, w) > 0$ on \mathcal{D} , $g(p, w) = 0$ on $\partial \mathcal{D}$ and $|\partial_w g(p, w)| = -\frac{1}{2} \frac{\partial g}{\partial n_w}(p, w) > 0$ on $\partial \mathcal{D}$, we can use $-g(p, w)$ as a defining function for \mathcal{D} and hence

$$\frac{\partial g}{\partial p_\gamma}(p, w) = -k_1^\gamma(p, w) |\partial_w g(p, w)|$$

and

$$\mathcal{L}^\gamma g(p, w) = -k_2^\gamma(p, w) |\partial_w g(p, w)|^3$$

for all $(p, w) \in \partial \mathcal{D}$. Since $g(p, w)$ is of class C^4 up to $\partial D(p)$, $\Delta_w g(p, w) = 0$ for $w \in \partial D(p)$ and hence from (2.6)

$$\begin{aligned} \frac{\partial^2 g}{\partial p_\gamma \partial \bar{p}_\gamma} &= -k_2^\gamma |\partial_w g| + 2\Re \left(\frac{\frac{\partial g}{\partial p_\gamma}}{|\partial_w g|} \sum_{\alpha=1}^n \frac{\frac{\partial g}{\partial \bar{w}_\alpha}}{|\partial_w g|} \frac{\partial^2 g}{\partial w_\alpha \partial \bar{p}_\gamma} \right) \\ &= -k_2^\gamma |\partial_w g| - 2\Re \left(k_1^\gamma \sum_{i=1}^n \frac{\frac{\partial g}{\partial \bar{w}_\alpha}}{|\partial_w g|} \frac{\partial^2 g}{\partial w_\alpha \partial \bar{p}_\gamma} \right) \end{aligned}$$

for $w \in \partial D(p)$. We summarize this in the following

Proposition 2.1. *The function $g(p, w)$ is smooth upto $\mathcal{D} \cup \partial \mathcal{D} = \{(p, w) : p \in D, w \in \overline{D}(p)\}$. If $1 \leq \gamma \leq n$ and $p \in D$, then*

- (1) $\frac{\partial g}{\partial p_\gamma}(p)$ is a harmonic function of $w \in D(p)$ with

$$\frac{\partial g}{\partial p_\gamma}(p, 0) = \frac{\partial \lambda}{\partial p_\gamma}(p)$$

and with boundary values

$$\frac{\partial g}{\partial p_\gamma}(p, w) = -k_1(p, w) |\partial_w g(p, w)|, \quad w \in \partial D(p),$$

- (2) $\frac{\partial^2 g}{\partial p_\gamma \partial \bar{p}_\gamma}(p)$ is a harmonic function of $w \in D(p)$ with

$$\frac{\partial^2 g}{\partial p_\gamma \partial \bar{p}_\gamma}(p, 0) = \frac{\partial^2 \lambda}{\partial p_\gamma \partial \bar{p}_\gamma}(p)$$

and with boundary values

$$\frac{\partial^2 g}{\partial p_\gamma \partial \bar{p}_\gamma}(p, w) = -k_2^\gamma(p, w) |\partial_w g(p, w)| - 2\Re \left(k_1^\gamma(p, w) \sum_{\alpha=1}^n \frac{\frac{\partial g}{\partial \bar{w}_\alpha}(p, w)}{|\partial_w g(p, w)|} \frac{\partial^2 g}{\partial w_\alpha \partial \bar{p}_\gamma}(p, w) \right), \quad w \in \partial D(p).$$

To this end, it was proved in [7] that $g(p, w)$ is C^2 up to $\{(p, w) : p \in \overline{D}, w \in \overline{D}(p)\}$ by deriving the following estimates: there exists a constant C independent of $p \in \partial D$ such that

$$(2.7) \quad \begin{cases} |k_1^\gamma(p, w)| \leq C|w|^2 \\ |k_2^\gamma(p, w)| \leq C|w|^3 \\ |\partial_w g(p, w)| \leq C|w|^{-2n+1} \\ \left| \frac{\partial^2 g}{\partial \bar{w}_\alpha \partial p_\gamma} \right| \leq C|w|^{-2n+2} \end{cases}$$

for all $w \in \partial D^\nu$ with $|w| \geq 1$. Moreover, the derivatives $\frac{\partial g}{\partial p_\gamma}$ and $\frac{\partial^2 g}{\partial p_\gamma \partial \bar{p}_\gamma}$ are given by the following variation formulae:

Proposition 2.2. *Let $1 \leq \gamma \leq n$. Then for $p \in \overline{D}$ and $a \in D(p)$,*

$$(2.8) \quad \frac{\partial g}{\partial p_\gamma}(p, a) = \frac{1}{2(n-1)\sigma_{2n}} \int_{\partial D(p)} k_1^\gamma(p, w) |\partial_w g(p, w)| \frac{\partial g_a(p, w)}{\partial n_w} dS_w$$

and

$$(2.9) \quad \begin{aligned} \frac{\partial^2 g}{\partial p_\gamma \partial \overline{p}_\gamma}(p, a) &= \frac{1}{2(n-1)\sigma_{2n}} \int_{\partial D(p)} k_2^\gamma(p, w) |\partial_w g(p, w)| \frac{\partial g_a(p, w)}{\partial n_w} dS_w \\ &\quad + \frac{1}{(n-1)\sigma_{2n}} \Re \sum_{\alpha=1}^n \int_{\partial D(p)} k_1^\gamma(p, w) \frac{\frac{\partial g}{\partial \overline{w}_\alpha}(p, w)}{|\partial_w g(p, w)|} \frac{\partial^2 g}{\partial w_\alpha \partial \overline{p}_\gamma}(p, w) \frac{\partial g}{\partial n_w}(w) dS_w. \end{aligned}$$

where $g_a(p, w)$ is the Green function for $D(p)$ with pole at a .

We note that for $p \in D$, the above formulae are consequences of proposition 2.1. For $p \in \partial D$, these formulae were obtained in [7] by finding

$$\lim_{D \ni q \rightarrow p} \frac{\partial g}{\partial p_\gamma}(q, a) \quad \text{and} \quad \lim_{D \ni q \rightarrow p} \frac{\partial^2 g}{\partial p_\gamma \partial \overline{p}_\gamma}(q, a).$$

A particular case of this proposition is the following:

Proposition 2.3. *Let $1 \leq \gamma \leq n$ and $p \in \overline{D}$.*

$$(2.10) \quad \frac{\partial \lambda}{\partial p_\gamma}(p) = \frac{1}{2(n-1)\sigma_{2n}} \int_{\partial D(p)} k_1^\gamma(p, \zeta) |\partial_w g(p, \zeta)| \frac{\partial g(p, w)}{\partial n_w} dS_w$$

and

$$(2.11) \quad \begin{aligned} \frac{\partial^2 \lambda}{\partial p_\gamma \partial \overline{p}_\gamma}(p) &= \frac{1}{2(n-1)\sigma_{2n}} \int_{\partial D(p)} k_2^\gamma(p, w) |\partial_w g(p, \zeta)|^2 dS_w \\ &\quad + \frac{1}{(n-1)\sigma_{2n}} \Re \sum_{\alpha=1}^n \int_{\partial D(p)} k_1^\gamma(p, w) \frac{\frac{\partial g}{\partial \overline{w}_\alpha}(p, w)}{|\partial_w g(p, w)|} \frac{\partial^2 g}{\partial w_\alpha \partial \overline{p}_\gamma}(p, w) \frac{\partial g}{\partial n_w}(p, w) dS_w. \end{aligned}$$

We now consider a sequence $\{D_\nu\}$ of C^∞ -smoothly bounded domains in \mathbf{C}^n with C^∞ -smooth defining functions ψ_ν such that $\{\psi_\nu\}$ converges in the C^∞ -topology on compact subsets of \mathbf{C}^n to ψ . In other words, $\{D_\nu\}$ converges in the C^∞ -topology to D . Another commonly used terminology for this is that the sequence $\{D_\nu\}$ is a C^∞ -perturbation of D . This implies, in particular, that D_ν converges in the Hausdorff sense to D . Now for each $\nu \geq 1$, consider the scaling map $T_\nu : D_\nu \times \mathbf{C}^n \rightarrow \mathbf{C}^n$ defined by

$$T_\nu(p, z) = \frac{z - p}{-\psi_\nu(p)}$$

and the family of domains $\{D_\nu(p) : p \in \overline{D}_\nu\}$ defined by

$$D_\nu(p) = \begin{cases} T_\nu(p, D_\nu) & ; \quad \text{if } p \in D_\nu, \\ \left\{ w \in \mathbf{C}^n : 2\Re \left(\sum_{i=1}^n \psi_{\nu i}(p) w_i \right) - 1 < 0 \right\} & ; \quad \text{if } p \in \partial D. \end{cases}$$

The normalised Robin function $\lambda_\nu(p)$ for (D_ν, ψ_ν) is then the Robin constant for $D_\nu(p)$ at 0. We will denote the Green function for D_ν with pole at 0 by $g_\nu(p, w)$. Also, let

$$\mathcal{D}_\nu = \cup_{p \in D_\nu} (p, D_\nu(p)) = \{(p, w) : p \in D_\nu, w \in D_\nu(p)\}$$

be the variation associated to (D_ν, ψ_ν) and let

$$(2.12) \quad f_\nu(p, w) = 2\Re \left\{ \sum_{\alpha=1}^n \int_0^1 \left(w_\alpha (\psi_\nu)_\alpha (p - \psi_\nu(p)tw) \right) dt \right\} - 1.$$

Then $f_\nu(p, w)$ is a smooth function on $\mathbf{C}^n \times \mathbf{C}^n$ that defines the variation \mathcal{D}_ν . It is evident that the functions $f_\nu(p, w)$ converge in the C^∞ -topology on compact subsets of $\mathbf{C}^n \times \mathbf{C}^n$ to the function

$$f(p, w) = 2\Re \left\{ \sum_{\alpha=1}^n \int_0^1 \left(w_\alpha \psi_\alpha (p - \psi(p)tw) \right) dt \right\} - 1$$

which defines the variation \mathcal{D} associated to (D, ψ) .

Now let $p_\nu \in D_\nu$ be such that $\{p_\nu\}$ converges to $p_0 \in \partial D$. For brevity, we let

$$(2.13) \quad \begin{aligned} T^\nu(z) &= T_\nu(p_\nu, z) = \frac{z - p_\nu}{-\psi_\nu(p_\nu)}, \\ D^\nu &= D_\nu(p_\nu) = T^\nu(D_\nu), \text{ and} \\ g^\nu(w) &= g_\nu(p_\nu, w). \end{aligned}$$

Thus $g^\nu(w)$ is the Green function for D^ν with pole at 0. Let $1 \leq \gamma \leq n$. By proposition 2.1, $\frac{\partial g_\nu}{\partial p_\gamma}(p_\nu, w)$ is a harmonic function of $w \in D^\nu$ with boundary values

$$(2.14) \quad -k_1^{\nu\gamma}(w)|\partial_w g^\nu(w)|$$

where

$$(2.15) \quad k_1^{\nu\gamma}(w) = k_{1\nu}^\gamma(w) = \frac{\partial f_\nu}{\partial p_\gamma}(p_\nu, w)|\partial_w f_\nu(p_\nu, w)|^{-1}.$$

Similarly, $\frac{\partial^2 g_\nu}{\partial p_\gamma \partial \bar{p}_\gamma}(p_\nu, w)$ is a harmonic function of $w \in D^\nu$ with boundary values

$$(2.16) \quad \frac{\partial^2 g_\nu}{\partial p_\gamma \partial \bar{p}_\gamma}(p_\nu, w) = -k_2^{\nu\gamma}(w)|\partial_w g^\nu(w)| - 2\Re\left(k_1^{\nu\gamma}(w) \sum_{\alpha=1}^n \frac{\frac{\partial g^\nu}{\partial \bar{w}_\alpha}(w)}{|\partial_w g^\nu(w)|} \frac{\partial^2 g_\nu}{\partial w_\alpha \partial \bar{p}_\gamma}(p_\nu, w)\right), \quad w \in \partial D^\nu$$

where

$$(2.17) \quad k_2^{\nu\gamma}(w) = \mathcal{L}^\gamma f_\nu(p_\nu, w)|\partial_w f_\nu(p_\nu, w)|^{-3}$$

and \mathcal{L}^γ is defined by (2.6). We want to conclude this section by finding uniform bounds for the functions $k_1^{\nu\gamma}(w)$ and $k_2^{\nu\gamma}(w)$ near the boundary of ∂D^ν which will be required to estimate the boundary values (2.14) and (2.16) in section 4 and 5. For $0 < r < 1$ let $\mathcal{E}^\nu(r)$ be the collar about ∂D^ν defined by

$$\mathcal{E}^\nu(r) = \cup_{w_0 \in \partial D^\nu} \{w \in D^\nu : |w - w_0| < r|w_0|\}.$$

Note that $\mathcal{E}^\nu(r)$ lies in D^ν and $\bar{\mathcal{E}}^\nu(r)$ does not contain the origin. Similarly, let $\mathcal{E}_\nu(r)$ be the collar around ∂D_ν defined by

$$\mathcal{E}_\nu(r) = \cup_{z_0 \in \partial D_\nu} \{z \in D_\nu : |z - z_0| < r|z_0 - p_\nu|\}.$$

Note that $\mathcal{E}_\nu(r)$ lies in D_ν and does not contain the point p_ν . Also, note that

$$(2.18) \quad \mathcal{E}_\nu(r) = (T^\nu)^{-1}(\mathcal{E}^\nu(r)).$$

Lemma 2.4. *There exists a constant $m > 0$, a number $0 < r < 1$, and an integer I such that*

$$|\partial_w f_\nu(p_\nu, w)| > m$$

for all $\nu \geq I$ and $w \in \mathcal{E}^\nu(r)$.

Proof. Choose a δ -neighbourhood U of ∂D i.e.,

$$U = \{z \in \mathbf{C}^n : d(z, \partial D) < \delta\}$$

and a constant $m > 0$ such that $|\partial\psi(p)| > 2m$ for $p \in U$. Since $\partial\psi_\nu$ converges uniformly on \bar{U} to $\partial\psi$, there exists an integer I such that

$$(2.19) \quad |\partial\psi_\nu(p)| > m$$

for $\nu \geq I$ and $p \in U$. Modify the integer I so that $\partial D_\nu \subset N(\delta/2)$ for all $\nu \geq I$. Since $p_\nu \rightarrow p_0 \in \partial D$, without loss of generality let us assume that $p_\nu \in U$ for all $\nu \geq I$. Now define

$$r = \frac{\delta}{3\delta + 2\text{diam}(D)}.$$

Then it is evident that

$$(2.20) \quad \mathcal{E}_\nu(r) \subset U$$

for $\nu \geq I$. Now fix $\nu \geq I$ and $w \in \mathcal{E}^\nu(r)$. If we define $z = T_\nu^{-1}w = p_\nu - \psi_\nu(p_\nu)w$ then, by (2.18)

$$z \in \mathcal{E}_\nu(r) \subset U.$$

From (2.12),

$$|\partial_w f_\nu(p_\nu, w)| = |\partial\psi_\nu(z)| > m$$

by (2.19). □

We now modify step 4 of chapter 4 [7] to obtain the following estimates:

Lemma 2.5. *Let r and I be as in lemma 2.4. Then there exists a constant $M > 0$ such that*

- (i) $|(\partial f_\nu / \partial w_\alpha)(p_\nu, w)| < M,$
- (ii) $|(\partial f_\nu / \partial p_\gamma)(p_\nu, w)| < M(1 + |w|^{-1})|w|^2,$
- (iii) $|(\partial^2 f_\nu / \partial w_\alpha \partial w_\beta)(p_\nu, w)| < M|w|^{-1},$
- (iv) $|(\partial^2 f_\nu / \partial p_\gamma \partial w_\alpha)(p_\nu, w)| < M(1 + |w|^{-1})|w|,$
- (v) $|(\partial^2 f_\nu / \partial p_\gamma \partial p_\mu)(p_\nu, w)| < M(1 + |w|^{-1} + |w|^{-2})|w|^3.$

for all $\nu \geq I$ and $w \in \mathcal{E}^\nu(r)$.

Proof. Let U be as in the proof of lemma 2.4 and choose $R > 0$ such that $U \subset B(0, R)$. Since $\{\psi_\nu\}$ converges in the C^∞ -topology on compact subsets of \mathbf{C}^n to ψ , we can find a constant $M_1 > 0$ such that $\psi, \psi_\nu, \nu \geq 1$, and their derivatives of order up to two are bounded in absolute value by M_1 on $\overline{B}(0, R)$.

Now let $\nu \geq I$ and let $w \in \mathcal{E}^\nu(r)$. Then we have

$$(2.21) \quad p_\nu - \psi_\nu(p_\nu)tw \in B(0, R), \quad 0 \leq t \leq 1.$$

Before proving this, note that this implies in particular that ψ_ν and its derivatives of order up to 2 are bounded in absolute value by M_1 at the points $p_\nu - \psi_\nu(p_\nu)tw$ for all $0 \leq t \leq 1$. Now to prove (2.21), let $0 \leq t \leq 1$. Let

$$z = T_\nu^{-1}w = p_\nu - \psi_\nu(p_\nu)w.$$

Then by (2.18) $z \in \mathcal{E}_\nu(r)$ and hence by (2.20), $z \in U$. Now

$$p_\nu - \psi_\nu(p_\nu)tw = p_\nu + t(z - p_\nu) = (1 - t)p_\nu + tz \in B(0, R)$$

as $p_\nu, z \in U \subset B(0, R)$.

(i) Differentiating (2.3) with respect to w_α under the integral sign, we have

$$\frac{\partial f}{\partial w_\alpha}(p, w) = \psi_\alpha(p - \psi(p)w), \quad p, w \in \mathbf{C}^n.$$

Hence for $\nu \geq I$ and $w \in \mathcal{E}^\nu(r)$,

$$\left| \frac{\partial f_\nu}{\partial w_\alpha}(p_\nu, w) \right| = |\psi_{\nu\alpha}(p_\nu - \psi_\nu(p_\nu)w)| \leq M_1.$$

(ii) Differentiating (2.3) with respect to p_γ under the integral sign, we have

$$\frac{\partial f}{\partial p_\gamma}(p, w) = \sum_{\alpha=1}^n \int_0^1 \frac{\partial}{\partial p_\gamma} \left(w_\alpha \psi_\alpha(p - \psi(p)tw) \right) + \frac{\partial}{\partial p_\gamma} \left(\overline{w}_\alpha \psi_{\overline{\alpha}}(p - \psi(p)tw) \right) dt, \quad p, w \in \mathbf{C}^n.$$

Observe that

$$\frac{\partial}{\partial p_\gamma} \left(w_\alpha \psi_\alpha(p - \psi(p)tw) \right) = w_\alpha \psi_{\gamma\alpha}(p - \psi(p)tw) - 2t\psi_\gamma(p)\Re \sum_{i=1}^n w_i w_\alpha \psi_{i\alpha}(p - \psi(p)tw).$$

Therefore,

$$(2.22) \quad \begin{aligned} \frac{\partial f}{\partial p_\gamma}(p, w) &= \sum_{\alpha=1}^n \int_0^1 \left(w_\alpha \psi_{\gamma\alpha}(p - \psi(p)tw) + \overline{w}_\alpha \psi_{\gamma\overline{\alpha}}(p - \psi(p)tw) \right) dt \\ &\quad - 2\psi_\gamma(p)\Re \sum_{i,\alpha=1}^n \int_0^1 \left(w_i w_\alpha \psi_{i\alpha}(p - \psi(p)tw) + w_i \overline{w}_\alpha \psi_{i\overline{\alpha}}(p - \psi(p)tw) \right) t dt. \end{aligned}$$

Hence, for $\nu \geq I$ and $w \in \mathcal{E}^\nu(r)$,

$$\begin{aligned} \left| \frac{\partial f_\nu}{\partial p_\gamma}(p_\nu, w) \right| &\leq \sum_{\alpha=1}^n \int_0^1 |w_\alpha| |\psi_{\nu\gamma\alpha}(p_\nu - \psi_\nu(p_\nu)tw)| + |\overline{w}_\alpha| |\psi_{\nu\gamma\overline{\alpha}}(p_\nu - \psi_\nu(p_\nu)tw)| dt \\ &\quad + 2|\psi_\nu(p_\nu)| \sum_{i,\alpha=1}^n \int_0^1 |w_i| |w_\alpha| |\psi_{\nu i\alpha}(p_\nu - \psi_\nu(p_\nu)tw)| + |w_i| |\overline{w}_\alpha| |\psi_{\nu i\overline{\alpha}}(p_\nu - \psi_\nu(p_\nu)tw)| t dt \\ &\leq \int_0^1 2|w| \sqrt{n} M_1 dt + 2M_1 \sum_{i=1}^n \int_0^1 2|w_i| |w| \sqrt{n} M_1 t dt \\ &\leq 2\sqrt{n} M_1 |w| + 2n^{3/2} (M_1)^2 |w|^2 \\ &\leq M_2 (1 + |w|^{-1}) |w|^2 \end{aligned}$$

where $M_2 = 2n^{3/2}(M_1)^2$.

(iii) Differentiating (2.3) with respect to w_α under the integral sign, we have

$$\frac{\partial f}{\partial w_\alpha}(p, w) = \psi_\alpha(p - \psi(p)w), \quad p, w \in \mathbf{C}^n.$$

Differentiating this equation with respect to w_β ,

$$\frac{\partial^2 f}{\partial w_\beta \partial w_\alpha}(p, w) = (-\psi(p))\psi_{\alpha\beta}(p - \psi(p)w), \quad p, w \in \mathbf{C}^n.$$

Let $\nu \geq I$ and $w \in \mathcal{E}^\nu(r)$. Let

$$z = T_\nu^{-1}w = p_\nu - \psi_\nu(p_\nu)w.$$

Then by (2.21), $z \in B(0, R)$. Now we have

$$\left| \frac{\partial^2 f_\nu}{\partial w_\beta \partial w_\alpha}(p_\nu, w) \right| \leq \frac{|z - p_\nu|}{|w|} |\psi_{\nu\alpha\beta}(z)| \leq 2RM_1|w|^{-1} = M_3|w|^{-1}$$

where $M_3 = 2RM_1$. Finally differentiating (2.22), we obtain (iv) and (v). \square

Proposition 2.6. *There exist $0 < r < 1$, a constant C and an integer I such that*

- (1) $|k_1^{\nu\gamma}(w)| \leq C(1 + |w|^{-1})|w|^2$, and
- (2) $|k_2^{\nu\gamma}(w)| \leq C(1 + |w|^{-1} + |w|^{-2})|w|^3$

for all $\nu \geq I$ and $w \in \bar{\mathcal{E}}^\nu(r)$.

Proof. Let $0 < r < 1$, $m > 0$ and I be as in lemma 2.4. Choose M as in lemma 2.5. Then from (2.15)

$$|k_1^\nu(w)| = \left| \frac{\partial f_\nu}{\partial p_\gamma}(p_\nu, w) \right| |\partial_w f_\nu(p_\nu, w)|^{-1} < \frac{M}{m} (1 + |w|^{-1})|w|^2$$

for $\nu \geq I$ and $w \in \mathcal{E}^\nu(r)$. Also, since $0 \notin \bar{\mathcal{E}}^\nu(r)$, the function

$$|k_1^\nu(w)|(1 + |w|^{-1})^{-1}|w|^{-2}$$

is continuous up to $\bar{\mathcal{E}}^\nu(r)$ and hence (1) follows.

Similarly, from (2.17)

$$\begin{aligned} |k_2^\nu(w)| &< \frac{1}{m^3} \left(M(1 + |w|^{-1} + |w|^{-2})|w|^3 M^2 + 2nM(1 + |w|^{-1})|w|^2 M M(1 + |w|^{-1})|w| \right. \\ &\quad \left. + (M(1 + |w|^{-1})|w|^2)^2 nM|w|^{-1} \right) \leq C(1 + |w|^{-1} + |w|^{-2})|w|^3 \end{aligned}$$

for some constant C whenever $\nu \geq I$ and $w \in \mathcal{E}^\nu(r)$. Again the function

$$|k_2^\nu(w)|(1 + |w|^{-1} + |w|^{-2})^{-1}|w|^{-3}$$

is continuous upto $\bar{\mathcal{E}}^\nu(r)$ and hence (2) follows. \square

3. ASYMPTOTICS OF Λ_ν

In this section we prove theorem 1.2. First, we recall the following stability result from [1].

Proposition 3.1. *Let D be a domain in \mathbf{C}^n with C^2 -smooth boundary and let $\{D_j\}$ be a C^2 -perturbation of D . Let $G(z, p)$ be the Green function for D with pole at p and let $\Lambda(p)$ be the Robin function for D . Similarly, let $G_j(z, p)$ be the Green function for D_j with pole at p and $\Lambda_j(p)$ the Robin function for D_j . Then*

$$\lim_{j \rightarrow \infty} G_j(z, p) = G(z, p)$$

uniformly on compact subsets of $D \setminus \{p\}$ and

$$\lim_{j \rightarrow \infty} D^{A\bar{B}} \Lambda_j(p) = D^{A, \bar{B}} \Lambda(p)$$

uniformly on compact subsets of D .

For a proof see [1, proposition 7.1, proposition 7.2]. This proposition, together with [7, proposition 5.1] yields the following boundary behaviour of the functions $G_j(z, p)$.

Corollary 3.2. *Let D be a domain in \mathbf{C}^n with C^∞ -smooth boundary and let $\{D_j\}$ be a C^∞ -perturbation of D . Let $z_j \in \overline{D}_j$ be such that $\{z_j\}$ converges to a point $z_0 \in \partial D$. Then for any $p \in D$,*

$$\lim_{j \rightarrow \infty} G_j(z_j, p) = G(z_0, p)$$

and identifying $z = (z_1, \dots, z_n) \in \mathbf{C}^n$ with $x = (x_1, \dots, x_{2n}) \in \mathbf{R}^{2n}$,

$$\lim_{j \rightarrow \infty} \frac{\partial G_j}{\partial x_k}(z_j, p) = \frac{\partial G}{\partial x_k}(z_0, p)$$

for $1 \leq k \leq 2n$.

Proof. Since the Green function is invariant under translation and rotation, without loss of generality, we assume that $z_0 = 0$ and the normal to ∂D at z_0 is along x_{2n} axis. By the implicit function theorem, we can find a ball $B(0, r)$, a C^∞ -smooth function ϕ defined on $B(0', r) \subset \mathbf{R}^{2n-1}$, a sequence $\{\phi_j\}$ of C^∞ -smooth functions defined on $B(0', r)$ that converges in C^∞ -topology on compact subsets of $B(0', r)$ to ϕ such that

$$(3.1) \quad \begin{cases} B(0, r) \cap \partial D = \{(x', \phi(x')) : x' \in B(0', r)\}, \\ B(0, r) \cap \partial D_j = \{(x', \phi_j(x')) : x' \in B(0', r)\}. \end{cases}$$

Now let $p \in D$. Shrinking r if necessary, let us assume that $2r < |p|$. Then for $z \in B(0, r) \cap D_j$,

$$(3.2) \quad G_j(z, p) < |z - p|^{-2n+2} < r^{-2n+2}.$$

Consider the dilation

$$Z = Sz = \frac{z}{r}$$

and set

$$\Omega = S(B(0, r) \cap D), \quad \Omega_j = S(B(0, r) \cap D_j).$$

Define

$$u(Z) = r^{2n-2} G(z, p), \quad Z \in \Omega,$$

and

$$u_j(Z) = r^{2n-2} G_j(z, p), \quad Z \in \Omega_j$$

Then by (3.1), (3.2) and in view of proposition 3.1, the sequence $\{u_j\}$ on $\{\Omega_j\}$ satisfies the hypothesis of [7, proposition 5.1] and therefore

$$\begin{cases} \lim_{j \rightarrow \infty} u_j(Z_j) = u(0), \\ \lim_{j \rightarrow \infty} \frac{\partial u_j}{\partial \bar{x}_k}(Z_j) = \frac{\partial u}{\partial \bar{x}_k}(0). \end{cases}$$

where $Z_j = Sz_j$. This implies that

$$\begin{cases} \lim_{j \rightarrow \infty} G_j(z_j, p) = G(0, p), \\ \lim_{j \rightarrow \infty} \frac{\partial G_j}{\partial x_k}(z_j, p) = \frac{\partial G}{\partial x_k}(0, p). \end{cases}$$

□

Proof of theorem 1.2. Consider the affine maps $T^\nu : \mathbf{C}^n \rightarrow \mathbf{C}^n$ defined by

$$T^\nu(z) = \frac{z - p_\nu}{-\psi_\nu(p_\nu)}$$

and the scaled domains $D^\nu = T^\nu(D_\nu)$. Recall from the previous section that a defining function for D^ν is given by

$$f_\nu(p_\nu, w) = 2\Re \left\{ \sum_{\alpha=1}^n \int_0^1 \left(w_\alpha \psi_{\nu\alpha}(p_\nu - \psi_\nu(p_\nu)tw) \right) dt \right\} - 1.$$

It is evident that $\{f_\nu(p_\nu, \cdot)\}$ converges in the C^∞ -topology on compact subsets of \mathbf{C}^n to

$$f(p_0, w) = 2\Re \left(\sum_{\alpha=1}^n \psi_\alpha(p_0) w_\alpha \right) - 1.$$

This implies that $\{D^\nu\}$ is a C^∞ -perturbation of the half space

$$\mathcal{H} = \left\{ w : 2\Re \left(\sum_{\alpha=1}^n \psi_\alpha(p_0) w_\alpha \right) - 1 < 0 \right\}.$$

Therefore, by proposition 3.1

$$(3.3) \quad \lim_{\nu \rightarrow \infty} D^{A\overline{B}} \Lambda_{D^\nu}(0) = D^{A\overline{B}} \Lambda_{\mathcal{H}}(0).$$

Now by [1, (1.1)],

$$\Lambda_{D^\nu}(p) = \Lambda_\nu(p_\nu - p\psi_\nu(p_\nu))(\psi_\nu(p))^{2n-2}$$

Differentiating this we obtain

$$D^{A\overline{B}} \Lambda_{D^\nu}(0) = (-1)^{|A|+|B|} D^{A\overline{B}} \Lambda_\nu(p_\nu)(\psi_\nu(p_\nu))^{2n-2+|A|+|B|}.$$

Hence from (3.3),

$$\lim_{\nu \rightarrow \infty} D^{A\overline{B}} (-1)^{|A|+|B|} D^{A\overline{B}} \Lambda_\nu(p_\nu)(\psi_\nu(p_\nu))^{2n-2+|A|+|B|} = D^{A\overline{B}} \Lambda_{\mathcal{H}}(0).$$

which completes the proof. \square

4. ESTIMATES ON THE FIRST DERIVATIVES

Let $1 \leq \gamma \leq n$. By proposition 2.1, $\frac{\partial g_\nu}{\partial p_\gamma}(p_\nu, w)$ is a harmonic function of $w \in D^\nu$,

$$\frac{\partial g_\nu}{\partial p_\gamma}(p_\nu, 0) = \frac{\partial \lambda_\nu}{\partial p_\gamma}(p_\nu)$$

and

$$(4.1) \quad \frac{\partial g_\nu}{\partial p_\gamma}(p_\nu, w) = -k_1^{\nu\gamma}(w) |\partial_w g^\nu(w)|, \quad w \in \partial D^\nu.$$

Therefore,

$$(4.2) \quad \frac{\partial \lambda_\nu}{\partial p_\gamma}(p_\nu) = \frac{1}{2(n-1)\sigma_{2n}} \int_{\partial D^\nu} k_1^{\nu\gamma}(w) |\partial_w g^\nu(w)| \frac{\partial g^\nu}{\partial n_w}(w) dS_w.$$

Thus to find the limit of the above integrals, we need to estimate the boundary values (4.1). For this we modify Step 3 of chapter 4 [7].

Lemma 4.1. *There exists a number $0 < \rho < 1$ and an integer I such that for $\nu \geq I$ and $w_0 \in \partial D^\nu$, we can find a ball of radius $\rho|w_0|$ that is externally tangent to ∂D^ν at w_0 .*

Proof. Since D is bounded, we can find a ball $B(0, R)$ which contains D . Since $\{D_\nu\}$ converges in C^2 -topology to D , there exists an integer I such that $D_\nu \subset B(0, R)$ for all $\nu \geq I$. By implicit function theorem, there exists a number $\tilde{\rho}$ such that modifying I we can find for each $\nu \geq I$ and $z_0 \in \partial D_\nu$, a ball of radius $\tilde{\rho}$ that is externally tangent to ∂D_ν at z_0 . Now let $\nu \geq I$ and $w_0 \in \partial D^\nu$. Since D^ν is obtained from D_ν by means of a translation followed by dialation of factor $-\psi_\nu(p_\nu)$, it follows that we can find a ball of radius $\tilde{\rho}/(-\psi_\nu(p_\nu))$ that is externally tangent to ∂D^ν at w_0 . Also there exists $z_0 \in \partial D_\nu$ such that

$$w_0 = \frac{z_0 - p_\nu}{-\psi_\nu(p_\nu)}$$

which implies that

$$\frac{\tilde{\rho}}{-\psi_\nu(p_\nu)} = \frac{\tilde{\rho}|w_0|}{|z_0 - p_\nu|} \geq \frac{\tilde{\rho}}{2R}|w_0|.$$

Thus taking $\rho = \tilde{\rho}/2R$, it follows that the we can find a ball of radius $\rho|w_0|$ that is tangent to ∂D^ν at w_0 . \square

Proposition 4.2. *There exists an integer I and a constanct $C > 0$ such that*

$$|\partial_w g^\nu(w)| \leq C|w|^{-2n+1}$$

for all $\nu \geq I$ and $w \in \partial D^\nu$.

Proof. Choose $0 < \rho < 1$, an integer I and a constant C as in lemma 4.1. Let $\nu \geq I$ and $w_0 \in \partial D^\nu$. Let B be the ball of radius $\rho|w_0|$ that is externally tangent to ∂D^ν at w_0 . Let E be the ball centred at w_0 and of radius $\rho|w_0|$. Then $w \in E$ implies that

$$|w| > |w_0| - \rho|w_0| = (1 - \rho)|w_0|.$$

Therefore, for $w \in E \cap D^\nu$,

$$0 < g^\nu(w) \leq |w|^{-2n+2} < ((1 - \rho)|w_0|)^{-2n+2}.$$

By step 2 of chapter 4 [7], we have

$$|\partial_w g^\nu(w_0)| \leq c((1-\rho)|w_0|)^{-2n+2}(\rho|w_0|)^{-1}$$

where c does not depend on $g^\nu(w)$ or D^ν . Thus

$$|\partial_w g^\nu(w_0)| \leq C|w_0|^{-2n+1},$$

where $C = c\rho^{-1}(1-\rho)^{-2n+2}$ is independent of ν and $w_0 \in \partial D^\nu$. \square

Proposition 4.3. *There exists a constant $C > 0$ and an integer I such that*

$$\left| \frac{\partial g_\nu}{\partial p_\gamma}(p_\nu, w) \right| = |k_1^{\nu\gamma}(w)| |\partial_w g^\nu(w)| \leq C(1+|w|^{-1})|w|^{-2n+3}, \quad w \in \partial D^\nu$$

for all $\nu \geq I$.

Proof. By proposition 2.6, there exists a constant C and an integer I such that

$$|k_1^{\nu\gamma}(w)| \leq C(1+|w|^{-1})|w|^2, \quad w \in \partial D^\nu$$

for all $\nu \geq I$. In view of proposition 4.2, we can modify the constant C and the integer I so that

$$|\partial_w g^\nu(w)| \leq C|w|^{-2n+1}, \quad w \in \partial D^\nu$$

for all $\nu \geq I$. Hence, from (4.1),

$$\left| \frac{\partial g_\nu}{\partial p_\gamma}(p_\nu, w) \right| = |k_1^{\nu\gamma}(w)| |\partial_w g^\nu(w)| \leq C^2(1+|w|^{-1})|w|^{-2n+3}, \quad w \in \partial D^\nu$$

for all $\nu \geq I$. \square

Proposition 4.4. $\lim_{\nu \rightarrow \infty} \frac{\partial \lambda_\nu}{\partial p_\gamma}(p_\nu) = \frac{\partial \lambda}{\partial p_\gamma}(p_0)$.

Proof. In view of proposition 2.3 we have to prove that

$$(4.3) \quad \lim_{\nu \rightarrow \infty} \frac{1}{2(n-1)\sigma_{2n}} \int_{\partial D^\nu} k_1^{\nu\gamma}(w) |\partial_w g^\nu(w)| \frac{\partial g^\nu}{\partial n_w}(w) dS_w \\ = \frac{1}{2(n-1)\sigma_{2n}} \int_{\partial \mathcal{H}} k_1^\gamma(p_0, w) |\partial g(p_0, w)| \frac{\partial g}{\partial n_w}(p_0, w) dS_w.$$

where $\mathcal{H} = D(p_0)$. Let $R > 1$. Then the boundary surfaces $B(0, R) \cap \partial D^\nu$ converge to $B(0, R) \cap \mathcal{H}$ continuously in the sense that the unit normal vectors

$$\frac{\partial_w g^\nu(w)}{|\partial_w g^\nu(w)|} \rightarrow \frac{\partial g(p_0, w)}{|\partial_w g(p_0, w)|}$$

uniformly on compact sets, except at the corners $B(0, R) \cap \partial D^\nu$. Also, if $w^\nu \in \partial D^\nu$ and $\{w^\nu\}$ converges to $w^0 \in \partial \mathcal{H}$, then by definition

$$(4.4) \quad \lim_{\nu \rightarrow \infty} k_1^{\nu\gamma}(w^\nu) = k_1^\gamma(p_0, w^0)$$

and by corollary 3.2

$$(4.5) \quad \lim_{\nu \rightarrow \infty} \frac{\partial g^\nu}{\partial w_\alpha}(w^\nu) = \frac{\partial g}{\partial w_\alpha}(p_0, w^0)$$

for $1 \leq \alpha \leq n$. Hence,

$$(4.6) \quad \lim_{\nu \rightarrow \infty} \frac{1}{2(n-1)\sigma_{2n}} \int_{B(0, R) \cap \partial D^\nu} k_1^{\nu\gamma}(w) |\partial_w g^\nu(w)| \frac{\partial g^\nu}{\partial n_w}(w) dS_w \\ = \frac{1}{2(n-1)\sigma_{2n}} \int_{B(0, R) \cap \partial \mathcal{H}} k_1^\gamma(p_0, w) |\partial g(p_0, w)| \frac{\partial g}{\partial n_w}(p_0, w) dS_w.$$

To estimate these integrals outside the ball $B(0, R)$, note that by proposition 4.3, there exists a constant C and an integer I such that

$$|k_1^{\nu\gamma}(w)| |\partial_w g^\nu(w)| \leq C|w|^{-2n+3}, \quad w \in \partial D^\nu, |w| > 1$$

for all $\nu \geq I$. Therefore,

$$(4.7) \quad \left| \frac{1}{2(n-1)\sigma_{2n}} \int_{B^c(0,R) \cap \partial D^\nu} k_1^{\nu\gamma}(w) |\partial_w g^\nu(w)| \frac{\partial g^\nu}{\partial n_w}(w) dS_w \right| \\ \leq CR^{-2n+3} \frac{1}{2(n-1)\sigma_{2n}} \int_{\partial B^c(0,R) \cap \partial D^\nu} \left(-\frac{\partial g^\nu}{\partial n_\zeta}(w) \right) dS_w$$

for all $\nu \geq I$. Since

$$\int_{\partial B^c(0,R) \cap \partial D^\nu} \left(-\frac{\partial g^\nu}{\partial n_\zeta}(w) \right) dS_w \leq \int_{\partial D^\nu} \left(-\frac{\partial g^\nu}{\partial n_w}(w) \right) dS_w = (2n-2)\sigma_{2n},$$

we have from (4.7)

$$(4.8) \quad \left| \frac{1}{2(n-1)\sigma_{2n}} \int_{B^c(0,R) \cap \partial D^\nu} k_1^{\nu\gamma}(w) |\partial_w g^\nu(w)| \frac{\partial g^\nu}{\partial n_w}(w) dS_w \right| = O(R^{-2n+3})$$

uniformly for all $\nu \geq I$. By (2.7), we can modify the constant C so that

$$|k_1^\gamma(p_0, w)| |\partial_w g(p_0, w)| \leq C|w|^{-2n+3}, \quad w \in \partial \mathcal{H}, |w| > 1$$

and as above we obtain

$$(4.9) \quad \left| \frac{1}{2(n-1)\sigma_{2n}} \int_{B^c(0,R) \cap \partial \mathcal{H}} k_1^\gamma(p_0, w) |\partial_w g(p_0, w)| \frac{\partial g}{\partial n_w}(w) dS_w \right| = O(R^{-2n+3}).$$

Now (4.3) follows from (4.6), (4.8) and (4.9). \square

Remark 4.5. Note that the arguments of this section also imply that for any $a \in \mathcal{H}$,

$$\lim_{\nu \rightarrow \infty} \frac{\partial g_\nu}{\partial p_\gamma}(p_\nu, a) = \lim_{\nu \rightarrow \infty} \frac{1}{2(n-1)\sigma_{2n}} \int_{\partial D^\nu} k_1^{\nu\gamma}(w) |\partial_w g^\nu(w)| \frac{\partial g_{\nu a}}{\partial n_w}(p_\nu, w) dS_w \\ = \frac{1}{2(n-1)\sigma_{2n}} \int_{\partial \mathcal{H}} k_1^\gamma(w) |\partial_w g^0(w)| \frac{\partial g_a}{\partial n_w}(p_\nu, w) dS_w = \frac{\partial g}{\partial p_\gamma}(p_0, a).$$

Moreover, by proposition 4.3, the functions $\frac{\partial g_\nu}{\partial p_\gamma}(p_\nu, w)$ are uniformly bounded on compact subsets of \mathcal{H} for all large ν . Indeed, let $\overline{B}(0, r) \subset \mathcal{H}$. Then $\overline{B}(0, r) \subset D^\nu$ for all large ν . It follows that

$$\left| \frac{\partial g_\nu}{\partial p_\gamma}(p_\nu, w) \right| \leq Cr^{-2n+3}(1+r^{-1})$$

for $w \in \partial D^\nu$ and hence for $w \in D^\nu$ by the maximum principle. Therefore, $\left\{ \frac{\partial g_\nu}{\partial p_\gamma}(p_\nu, a) \right\}$ converges uniformly on compact subsets of \mathcal{H} to $\frac{\partial g}{\partial p_\gamma}(p_0, a)$.

5. ESTIMATES ON THE SECOND DERIVATIVES

By proposition 2.1, $\frac{\partial^2 g_\nu}{\partial p_\gamma \partial \overline{p}_\gamma}(p_\nu, w)$ is a harmonic function of $w \in D^\nu$,

$$\frac{\partial^2 g_\nu}{\partial p_\gamma \partial \overline{p}_\gamma}(p_\nu, 0) = \frac{\partial^2 \lambda_\nu}{\partial p_\gamma \partial \overline{p}_\gamma}(p_\nu),$$

and

$$(5.1) \quad \frac{\partial^2 g_\nu}{\partial p_\gamma \partial \overline{p}_\gamma}(p_\nu, w) = -k_2^{\nu\gamma}(w) |\partial_w g^\nu(w)| - 2\Re \left(k_1^{\nu\gamma}(w) \sum_{\alpha=1}^n \frac{\frac{\partial g^\nu}{\partial \overline{w}_\alpha}(w)}{|\partial_w g^\nu(w)|} \frac{\partial^2 g_\nu}{\partial w_\alpha \partial \overline{p}_\gamma}(p_\nu, w) \right), \quad w \in \partial D^\nu.$$

Therefore,

$$(5.2) \quad \frac{\partial^2 \lambda_\nu}{\partial p_\gamma \partial \overline{p}_\gamma}(p_\nu) = \frac{1}{2(n-1)\sigma_{2n}} \int_{\partial D^\nu} k_2^\nu(w) |\partial_w g^\nu(\zeta)| \frac{\partial g^\nu}{\partial n_w}(w) dS_w \\ + \frac{1}{(n-1)\sigma_{2n}} \Re \sum_{\alpha=1}^n \int_{\partial D^\nu} k_1^{\nu\gamma}(w) \frac{\frac{\partial g^\nu}{\partial \overline{w}_\alpha}(w)}{|\partial_w g^\nu(w)|} \frac{\partial^2 g_\nu}{\partial w_\alpha \partial \overline{p}_\gamma}(p_\nu, w) \frac{\partial g^\nu}{\partial n_w}(w) dS_w.$$

By similar arguments as in the previous section

$$(5.3) \quad \lim_{\nu \rightarrow \infty} \frac{1}{2(n-1)\sigma_{2n}} \int_{\partial D^\nu} k_2^\nu(w) |\partial_w g^\nu(\zeta)| \frac{\partial g^\nu}{\partial n_w}(w) dS_w \\ = \frac{1}{2(n-1)\sigma_{2n}} \int_{\partial \mathcal{H}} k_2(p_0, w) |\partial_w g(p_0, w)| \frac{\partial g}{\partial n_w}(w) dS_w$$

where $\mathcal{H} = D(p_0)$. Thus we only need to find the limit of the second integrals. This requires to estimate the functions

$$(5.4) \quad \frac{\partial^2 g_\nu}{\partial w_\alpha \partial \bar{p}_\gamma}(p_\nu, w)$$

on ∂D^ν . Since $\frac{\partial g_\nu}{\partial p_\gamma}(p_\nu, w)$ is a harmonic function of $w \in D^\nu$ with boundary values

$$(5.5) \quad F^\nu(w) = -k_1^{\nu\gamma}(w) |\partial_w g^\nu(w)| = -\frac{\frac{\partial f_\nu}{\partial p_\gamma}(p_\nu, w)}{|\partial_w f_\nu(p_\nu, w)|} |\partial_w g^\nu(w)|,$$

to estimate (5.4), we need to estimate the derivatives of $F^\nu(w)$. This will be done by modifying Steps 2 and 3 of chapter 5 [7].

In what follows we will identify the point $z = (z_1, \dots, z_n)$ in \mathbf{C}^n with the point $x = (x_1, \dots, x_{2n})$ in \mathbf{R}^{2n} . Similarly $w = (w_1, \dots, w_n)$ and $W = (W_1, \dots, W_n)$ in \mathbf{C}^n will be identified with $y = (y_1, \dots, y_{2n})$ and $Y = (Y_1, \dots, Y_{2n})$ in \mathbf{R}^{2n} respectively. First, we note the following version of a tubular neighbourhood theorem:

Proposition 5.1. *There exist $0 < r < 1$ and $M > 1$ and an integer I such that for $\nu \geq I$ and any $z_0 = (x'_0, x_{02n})$ in the neighbourhood*

$$\bigcup_{z \in \partial D_\nu} \{z + tn_z : -r < t < r\}$$

of ∂D_ν , $B(z_0, r) \cap \partial D_\nu$ can be represented, after a rotation and translation of coordinates, in the form $x_{2n} = \phi(x')$ where

- (a) $\phi(x')$ is smooth in $B(x'_0, r) \subset \mathbf{R}^{2n-1}$ with $\phi(x'_0) = x_{02n} - t$, where t is such that $z_0 = z_0^* + tn_{z_0^*}$ for some $z_0^* \in \partial D^\nu$, and
- (b) all partial derivatives of ϕ of order upto 6 are bounded in absolute value on $B(x'_0, r)$ by M .

Now fix r , M and I as in proposition 5.1. Modifying the integer I , if necessary, we may assume that

$$d(p_\nu, \partial D) < r$$

and

$$\partial D_\nu \subset \{z : d(z, \partial D) < r\}$$

for all $\nu \geq I$. This would imply that

$$(5.6) \quad |\tilde{z}_\nu - p_\nu| < \text{diam}(D) + 2r$$

for $\nu \geq I$ and $\tilde{z}_\nu \in \partial D_\nu$. Now, choose $0 < \eta < 1$ such that

$$(5.7) \quad \frac{\eta}{1-\eta} (\text{diam}(D) + 2r) < r.$$

Lemma 5.2. *Let $\nu \geq I$ and $w^\nu \in D^\nu \setminus \{0\}$ be such that*

$$\{w \in \mathbf{C}^n : |w - w^\nu| < \eta |w^\nu|\} \cap \partial D^\nu \neq \emptyset.$$

Let $S^\nu : \mathbf{C}^n \rightarrow \mathbf{C}^n$ be the affine map defined by

$$W = S^\nu(w) = \frac{w - w^\nu}{\eta |w^\nu|}$$

and set

$$\Omega^\nu = S^\nu \left(\{w \in \mathbf{C}^n : |w - w^\nu| < \eta |w^\nu|\} \cap D^\nu \right) = \{|W| < 1\} \cap S^\nu(D^\nu).$$

Then we can find $\Phi^\nu \in C^\infty(\{Y' : |Y'| < 1\})$ with

- (1) $\{|W| < 1\} \cap \partial \Omega^\nu = \{Y_{2n} = \Phi^\nu(Y')\}$, and
- (2) $\left| \frac{\partial^\alpha \Phi^\nu}{\partial Y^\alpha} \right| \leq M$ for $\alpha = (\alpha_1, \dots, \alpha_n)$ with $|\alpha| \leq 6$ if $|Y'| < 1$.

Proof. Let

$$z_\nu = (T^\nu)^{-1}(w^\nu) = p_\nu - \psi_\nu(p_\nu)w_\nu$$

and let

$$b_\nu = (T^\nu)^{-1}\left(\{w : |w - w^\nu| < \eta|w_\nu|\}\right) = \{z \in \mathbf{C}^n : |z - z_\nu| < \eta|z_\nu - p_\nu|\}.$$

Then $b_\nu \cap \partial D_\nu \neq \emptyset$ and hence there is a point $\tilde{z}_\nu \in \partial D_\nu$ such that

$$|\tilde{z}_\nu - z_\nu| < \eta|z_\nu - p_\nu| \leq \eta(|z_\nu - \tilde{z}_\nu| + |\tilde{z}_\nu - p_\nu|).$$

Therefore,

$$(5.8) \quad |\tilde{z}_\nu - z_\nu| < \frac{\eta}{1-\eta}|\tilde{z}_\nu - p_\nu| \leq \frac{\eta}{1-\eta}(\text{diam}(D) + 2r) < r$$

by (5.6) and (5.7) and hence

$$z_\nu \in \bigcup_{z \in \partial D_\nu} \{z + tn_z : -r < t < r\}.$$

By proposition 5.1, $B(z_\nu, r) \cap \partial D_\nu$ can be represented after a rotation and translation of coordinates in the form $x_{2n} = \phi_\nu(x')$ where $\phi_\nu(x')$ is C^∞ on $B(x'_\nu, r)$,

$$(5.9) \quad \phi_\nu(x'_\nu) = x_{\nu 0} - t_\nu$$

where

$$(5.10) \quad -t_\nu = d(z_\nu, \partial D_\nu) < \eta|z_\nu - p_\nu|$$

and all partial derivatives of ϕ_ν of order up to 6 are bounded in absolute value by M . The surface

$$\{(x', x_{2n}) : x_{2n} = \phi_\nu(x'), |x' - x'_\nu| < r\}$$

is mapped by $S^\nu \circ T^\nu$ onto the surface

$$\{(Y', Y_{2n}) : Y_{2n} = \Phi^\nu(Y'), |Y'| < R^\nu\}$$

where, letting $w^\nu = (y^{\nu'}, y_{2n}^\nu)$, $p_\nu = (p'_\nu, p_{\nu 2n})$

$$\Phi^\nu(Y') = \frac{\phi_\nu(p'_\nu - \psi_\nu(p_\nu)y^{\nu'} - \psi_\nu(p_\nu)\eta|w^\nu|Y')}{-\psi_\nu(p_\nu)\eta|w^\nu|} + \frac{\psi_\nu(p_\nu)y_{2n}^\nu - p_{\nu 2n}}{-\psi_\nu(p_\nu)\eta|w^\nu|}$$

and

$$R^\nu = \frac{r}{-\psi_\nu(p_\nu)\eta|w^\nu|} = \frac{r}{\eta|z_\nu - p_\nu|}.$$

But from (5.8)

$$\eta|z_\nu - p_\nu| \leq \eta(|z_\nu - \tilde{z}_\nu| + |\tilde{z}_\nu - p_\nu|) \leq \eta\left(\frac{\eta}{1-\eta}|\tilde{z}_\nu - p_\nu| + |\tilde{z}_\nu - p_\nu|\right) = \frac{\eta}{1-\eta}|\tilde{z}_\nu - p_\nu| < r$$

so that $R^\nu > 1$. This implies that

$$\{|W| < 1\} \cap \partial\Omega^\nu \subset \{(Y', Y_{2n}) : Y_{2n} = \Phi^\nu(Y'), |Y'| < R^\nu\}.$$

By using the properties of ϕ_ν and the explicit formula for Φ^ν above, it follows that

$$\{|W| < 1\} \cap \partial\Omega^\nu = \{Y_{2n} = \Phi^\nu(Y')\}$$

where $\Phi^\nu \in C^\infty(\{Y' : |Y'| < 1\})$ and satisfies

- (a) $0 < \Phi^\nu(0) < 1$ by (5.9) and (5.10), and
- (b) $|\frac{\partial^\alpha \Phi^\nu}{\partial Y^\alpha}| < M$ for all $\alpha = (\alpha_1, \dots, \alpha_n)$ with $|\alpha| \leq 6$ if $|Y'| < 1$.

□

Now we modify Step 2 of chapter 5 [7], to obtain the following uniform estimates:

Proposition 5.3. *There exists a constant $C > 0$ and an integer I such that for $1 \leq i, j, k \leq 2n$*

- (1) $|(\partial g^\nu / \partial y_i)(w)| \leq C|w|^{-2n+1}$,
- (2) $|(\partial^2 g^\nu / \partial y_i \partial y_j)(w)| \leq C|w|^{-2n}$,
- (3) $|(\partial^3 g^\nu / \partial y_i \partial y_j \partial y_k)(w)| \leq C|w|^{-2n-1}$

for all $\nu \geq I$ and $w \in \overline{D}^\nu \setminus \{0\}$.

Proof. The proofs for (1), (2) and (3) are similar and so we prove only (1). Fix $1 \leq i \leq 2n$. Suppose that (1) is not true. Then there exists a sequence $\{w^\nu\}$ such that $w^\nu \in D^\nu \setminus \{0\}$ and

$$(5.11) \quad \lim_{\nu \rightarrow \infty} \left| \frac{\partial g^\nu}{\partial y_i}(w^\nu) \right| |w^\nu|^{2n-1} = \infty.$$

We claim that for all but finitely many ν ,

$$B(w^\nu) = \{w \in \mathbf{C}^n : |w - w^\nu| < \eta |w^\nu|\}$$

intersects ∂D^ν . Indeed, suppose that $B(w^\nu) \cap \partial D^\nu = \emptyset$ for some ν . Then $B(w^\nu) \subset D^\nu$ and therefore,

$$g^\nu(w) \leq |w|^{-2n+2} \leq (1 - \eta)^{-2n+2} |w^\nu|^{-2n+2}, \quad w \in \partial B(w^\nu).$$

Now, by Poisson integral formula, there exists a constant $c_n > 0$ independent of ν such that

$$\left| \frac{\partial g^\nu}{\partial y_i}(w^\nu) \right| \leq \frac{c_n}{(1 - \eta)^{2n-2}\eta} |w^\nu|^{-2n+1}.$$

But this can be true only for finitely many ν by (5.11) and hence the claim. Therefore, if we let

$$\Omega^\nu = S^\nu(B(w^\nu) \cap D^\nu) = \{|W| < 1\} \cap S^\nu(D^\nu)$$

then by lemma 5.2, we can find for all large ν , functions $\Phi^\nu \in C^\infty(\{Y' : |Y'| < 1\})$ such that

$$\Omega^\nu = \{|W| < 1\} \cap \{Y = (Y', Y_{2n}) : |Y'| < 1, Y_{2n} < \Phi^\nu(Y')\}$$

and

$$\left| \frac{\partial^\alpha \Phi^\nu}{\partial Y^\alpha} \right| < M \text{ for all } |\alpha| \leq N, \text{ if } |Y'| < 1.$$

Since M is independent of ν , by the Arzela-Ascoli theorem, after passing to a subsequence, $\{\Phi^\nu\}$ together with all partial derivatives of order up to 6 converge uniformly on compact subsets of $\{Y' : |Y'| < 1\}$ to a function $\Phi \in C^6(\{Y' : |Y'| < 1\})$. Set

$$\Omega = \{|W| < 1\} \cap \{Y = (Y', Y_{2n}) : |Y'| < 1, Y_{2n} < \Phi(Y')\}.$$

Now define the function u^ν on Ω^ν by

$$u^\nu(W) = |w_\nu|^{2n-2} (1 - \eta)^{2n-2} g^\nu(w)$$

for $W = (w - w_\nu)/(\eta |w_\nu|)$. Then u^ν is harmonic on Ω^ν , continuous up to $\partial \Omega^\nu$, and $u_\nu(W) = 0$ on $\{|W| < 1\} \cap \partial \Omega^\nu$. Since

$$0 < g^\nu(w) < |w|^{-2n+2} < (1 - \eta)^{-2n+2} |w^\nu|^{-2n+2}, \quad w \in B(w^\nu) \cap D^\nu$$

we have

$$0 < u^\nu(W) < 1, \quad W \in \Omega^\nu$$

By Harnack's theorem, passing to a subsequence, $\{u^\nu\}$ converges uniformly on compact subsets of Ω to a harmonic function u on Ω . From [7, proposition 5.1], it follows that

$$\lim_{\nu \rightarrow \infty} \left| \frac{\partial u^\nu}{\partial y_i}(0) \right| = \frac{\partial u}{\partial y_i}(0)$$

which is finite. Hence from the definition of u^ν ,

$$\lim_{\nu \rightarrow \infty} \left| \frac{\partial g^\nu}{\partial y_i}(w^\nu) \right| |w^\nu|^{2n-1} < \infty$$

which is a contradiction. Hence (1) must hold. \square

We now want to modify Step 3 of chapter 4 [7]. Recall that

$$\mathcal{E}^\nu(r) = \bigcup_{w_0 \in \partial D^\nu} \{w \in D^\nu : |w - w_0| < r |w_0|\}$$

is a collar about ∂D^ν lying in D^ν whose closure does not contain the origin. Similarly

$$\mathcal{E}_\nu(r) = (T^\nu)^{-1}(\mathcal{E}^\nu(r)) = \bigcup_{z_0 \in \partial D_\nu} \{z \in D_\nu : |z - z_0| < r_0 |z_0 - p_\nu|\}$$

is a collar about ∂D_ν lying in D_ν whose closure does not contain the point p_ν .

Lemma 5.4. *There exist $0 < r_0 < 1$, a constant $C > 0$ and an integer I such that*

$$(5.12) \quad \left| \frac{\partial^2 g^\nu}{\partial y_i \partial y_j}(w) \right| |\partial_w g^\nu(w)|^{-1} \leq C |w|^{-1}, \quad w \in \mathcal{E}^\nu(r_0)$$

for all $\nu \geq I$.

Proof. By the relation

$$g^\nu(w) = \psi_\nu(p_\nu)^{2n-2} G_\nu(z, p_\nu), \quad z = p_\nu - \psi_\nu(p_\nu)w,$$

we observe that (5.12) is equivalent to

$$(5.13) \quad \left| \frac{\partial^2 G_\nu}{\partial x_i \partial x_j}(z, p_\nu) \right| |\partial_z G_\nu(z, p_\nu)|^{-1} \leq C |z - p_\nu|^{-1}, \quad z \in \mathcal{E}_\nu(r_0).$$

We prove (5.13) by contradiction. So, suppose that there do not exist $0 < r_0 < 1$, $C > 0$ and integer I such that (5.13) holds for all $\nu \geq I$. Then there exist a sequence $\{z_{0\nu}\}$ with $z_{0\nu} \in \partial D_\nu$, and a sequence $\{z_\nu\}$ with

$$(5.14) \quad z_\nu \in D_\nu \text{ and } |z_\nu - z_{0\nu}| < \frac{1}{\nu} |z_{0\nu} - p_\nu|, \quad \nu \geq 1$$

such that

$$(5.15) \quad \left| \frac{\partial^2 G_\nu}{\partial x_i \partial x_j}(z_\nu, p_\nu) \right| |\partial_z G_\nu(z_\nu, p_\nu)|^{-1} \geq \nu |z_\nu - p_\nu|^{-1}, \quad \nu \geq 1.$$

By passing to a subsequence if necessary, we may assume that

$$\lim_{\nu \rightarrow \infty} z_{0\nu} = z_0 \in \partial D.$$

Then, from (5.14),

$$\lim_{\nu \rightarrow \infty} z_\nu = z_0.$$

Claim: $p_0 = z_0$. Suppose that this is not true. Then we can find an $\epsilon > 0$ such that $B(p_0, 2\epsilon) \cap B(z_0, \epsilon) = \emptyset$. Taking ϵ sufficiently small and ν sufficiently large, we can find by the implicit function theorem a C^∞ -smooth function ϕ on $B(x'_0, \epsilon)$ and a sequence $\{\phi_\nu\}$ of C^∞ -smooth functions on $B(x'_0, \epsilon)$ that converges in C^∞ -topology on compact subsets of $B(x'_0, \epsilon)$ to ϕ such that

$$(5.16) \quad \begin{cases} B(z_0, \epsilon) \cap \partial D = \{(x', \phi(x')) : x' \in B(x'_0, \epsilon)\}, \\ B(z_0, \epsilon) \cap \partial D_\nu = \{(x', \phi_\nu(x')) : x' \in B(x'_0, \epsilon)\}. \end{cases}$$

Without loss of generality let us assume that all p_ν lie in $B(p_0, \epsilon)$. Then

$$(5.17) \quad G_\nu(z, p_\nu) \leq |z - p_\nu|^{-2n+2} < \epsilon^{-2n+2}, \quad z \in B(z_0, \epsilon) \cap D_\nu$$

Now consider the affine map

$$Z = Sz = \frac{z - z_0}{\epsilon}$$

and set

$$\Omega = S(B(z_0, \epsilon/2) \cap D), \quad \Omega_\nu = S(B(z_0, \epsilon/2) \cap D_\nu).$$

Define

$$h_\nu(Z) = \epsilon^{2n-2} G_\nu(z, p_\nu), \quad Z \in \Omega_\nu.$$

Then h_ν is harmonic on Ω_ν , $h_\nu = 0$ on $B(0, 1) \cap \partial \Omega_\nu$ and by (5.17)

$$0 < h_\nu(Z) \leq 1, \quad Z \in \Omega_\nu.$$

Therefore, by Harnack's principle, after passing to a subsequence if necessary, $\{h_\nu\}$ converges uniformly on compact subsets of Ω to a positive harmonic function h . In view of (5.16), the sequence $\{h_\nu\}$ on $\{\Omega_\nu\}$ satisfies the hypothesis of [7, proposition 5.1] and hence

$$(5.18) \quad \begin{cases} \lim_{\nu \rightarrow \infty} |\partial_Z h_\nu(Z_\nu)| = |\partial_Z h(0)|, \\ \lim_{\nu \rightarrow \infty} \left| \frac{\partial^2 h_\nu}{\partial X_i \partial X_j}(Z_\nu) \right| = \left| \frac{\partial^2 h}{\partial X_i \partial X_j}(0) \right| < \infty \end{cases}$$

where $Z_\nu = Sz_\nu$. By the Hopf lemma,

$$|\partial_Z h(0)| > 0.$$

Hence

$$\lim_{\nu \rightarrow \infty} \frac{\left| \frac{\partial^2 G_\nu}{\partial x_i \partial x_j}(z_\nu, p_\nu) \right|}{|\partial_z G_\nu(z_\nu, p_\nu)|} |z_\nu - p_\nu| = \epsilon \lim_{\nu \rightarrow \infty} \frac{\left| \frac{\partial^2 h_\nu}{\partial X_i \partial X_j}(Z_\nu) \right|}{|\partial_Z h_\nu(Z_\nu)|} |z_\nu - p_\nu| = \frac{\left| \frac{\partial^2 h}{\partial X_i \partial X_j}(0) \right|}{|\partial_Z h(0)|} |z_0 - p_0| < \infty$$

which contradicts (5.15). Therefore, we must have $p_0 = z_0$ and hence the claim.

Now define

$$k_\nu = |p_\nu - z_{0\nu}|.$$

Consider the affine maps $S_\nu : \mathbf{C}^n \rightarrow \mathbf{C}^n$ defined by

$$\tilde{z} = S_\nu(z) = \frac{z - p_\nu}{k_\nu}$$

and let $\tilde{D}_\nu = S_\nu(D_\nu)$. A defining function for \tilde{D}_ν is given by

$$\begin{aligned} \psi_\nu \circ S_\nu^{-1}(\tilde{z}) &= \psi_\nu(p_\nu + k_\nu \tilde{z}) \\ &= \psi_\nu(p_\nu) + 2k_\nu \Re\left(\sum_{\alpha=1}^n (\psi_\nu)_\alpha(p_\nu) \tilde{z}_\alpha\right) + k_\nu^2 O(1) \end{aligned}$$

for \tilde{z} on a compact subset of \mathbf{C}^n . Since $\{\psi_\nu\}$ converges in the C^∞ -topology on compact subsets of \mathbf{C}^n to ψ , we note that $O(1)$ is independent of ν . Now

$$\tilde{\psi}_\nu(\tilde{z}) = \frac{\psi_\nu \circ S_\nu^{-1}(\tilde{z})}{k_\nu} = \frac{\psi_\nu(p_\nu)}{k_\nu} + 2\Re\left(\sum_{\alpha=1}^n (\psi_\nu)_\alpha(p_\nu) \tilde{z}_\alpha\right) + k_\nu O(1)$$

is again a defining function for \tilde{D}_ν . Note that we can find a ball B centered at p_0 , positive smooth functions ϕ_ν on B such that

$$-\psi_\nu(p) = \phi_\nu(p) d(p, \partial D_\nu), \quad p \in B.$$

By differentiating the above relation, it can be seen that the functions ϕ_ν , for all large ν , are uniformly bounded above by a constant $c > 0$ on possibly a smaller ball B' centered at p_0 . This implies that for all large ν ,

$$\left| \frac{\psi_\nu(p_\nu)}{k_\nu} \right| \leq \frac{cd_\nu(p_\nu, \partial D_\nu)}{|p_\nu - z_{0\nu}|} \leq c$$

and hence after passing to a subsequence, $\{\psi_\nu(p_\nu)/k_\nu\}$ converges to a number $\tilde{c} \leq 0$. Thus the functions $\tilde{\psi}_\nu$ converge in the C^∞ -topology on compact subsets of \mathbf{C}^n to the function

$$\tilde{\psi}(\tilde{z}) = \tilde{c} + 2\Re\left(\sum_{\alpha=1}^n \psi_\alpha(p_0) \tilde{z}_\alpha\right).$$

This implies that the domains \tilde{D}_ν are C^∞ -perturbation of the half space

$$\tilde{H} = \{\tilde{z} \in \mathbf{C}^n : \tilde{c} + 2\Re\left(\sum_{\alpha=1}^n \psi_\alpha(p_0) \tilde{z}_\alpha\right) < 0\}.$$

Since $\tilde{c} \leq 0$, it is evident that

$$(5.19) \quad 0 \in \overline{\tilde{H}}.$$

We will now derive a contradiction by proving that (5.19) is false. First, observe that $0 = S_\nu(p_\nu) \in \tilde{D}_\nu$. Let $\tilde{g}_\nu(\tilde{z})$ be the Green function for \tilde{D}_ν with pole at 0. Then

$$(5.20) \quad \tilde{g}_\nu(\tilde{z}) = G(z, p_\nu) k_\nu^{2n-2}.$$

Now let $\tilde{z}_{0\nu} = S_\nu(z_{0\nu})$. Then $\tilde{z}_{0\nu} \in \partial \tilde{D}_\nu$ and

$$|\tilde{z}_{0\nu}| = \left| \frac{z_{0\nu} - p_\nu}{k_\nu} \right| = 1.$$

Therefore, after passing to a subsequence, $\{\tilde{z}_{0\nu}\}$ converges to a point \tilde{z}_0 with

$$|\tilde{z}_0| = 1.$$

Evidently, $\tilde{z}_0 \in \partial \tilde{H}$. Also, let $\tilde{z}_\nu = S_\nu(z_\nu)$. Then

$$|\tilde{z}_\nu - \tilde{z}_{0\nu}| = \left| \frac{z_\nu - z_{0\nu}}{k_\nu} \right| < \frac{1}{\nu}$$

by (5.14). Therefore,

$$\lim_{\nu \rightarrow \infty} \tilde{z}_\nu = \tilde{z}_0.$$

Now we derive the contradiction by considering the following two cases:

Case I. $0 \in \tilde{H}$. Let $\tilde{g}(\tilde{z})$ be the Green function for \tilde{H} with pole at 0. Then by corollary 3.2,

$$\begin{cases} \lim_{\nu \rightarrow \infty} |\partial_{\tilde{z}} \tilde{g}_\nu(\tilde{z}_\nu)| = |\partial_{\tilde{z}} \tilde{g}(\tilde{z}_0)| > 0, \\ \lim_{\nu \rightarrow \infty} \frac{\partial^2 \tilde{g}_\nu}{\partial \tilde{x}_k \partial \tilde{x}_l}(\tilde{z}_\nu) = \frac{\partial^2 \tilde{g}}{\partial \tilde{x}_k \partial \tilde{x}_l}(\tilde{z}_0) \neq \infty. \end{cases}$$

Now from (5.20),

$$\lim_{\nu \rightarrow \infty} \left| \frac{\partial^2 G}{\partial x_i \partial x_j}(z_\nu, p_\nu) \right| \left| \partial_z G(z_\nu, p_\nu) \right|^{-1} |z_\nu - p_\nu| = \lim_{\nu \rightarrow \infty} \left| \frac{\partial^2 \tilde{g}_\nu}{\partial \tilde{x}_i \partial \tilde{x}_j}(\tilde{z}_\nu) \right| \left| \partial_{\tilde{z}} \tilde{g}_\nu(\tilde{z}_\nu) \right|^{-1} |\tilde{z}_\nu| < \infty$$

which contradicts (5.15) and hence $0 \notin \tilde{\mathcal{H}}$.

Case II. $0 \in \partial \tilde{H}$. By the implicit function theorem we can find a ball $B(\tilde{z}_0, \epsilon)$, a C^∞ -smooth function ϕ on $B(\tilde{x}'_0, \epsilon)$ and a sequence $\{\phi_\nu\}$ of C^∞ -smooth functions on $B(\tilde{x}'_0, \epsilon)$ that converges in the C^∞ -topology on compact subsets of $B(\tilde{x}'_0, \epsilon)$ to ϕ such that

$$(5.21) \quad \begin{cases} B(\tilde{x}_0, \epsilon) \cap \partial \tilde{H} = \{(\tilde{x}', \phi(\tilde{x}')) : \tilde{x}' \in B(\tilde{x}'_0, \epsilon)\} \\ B(\tilde{x}_0, \epsilon) \cap \partial \tilde{D}_\nu = \{(\tilde{x}', \phi_\nu(\tilde{x}')) : \tilde{x}' \in B(\tilde{x}'_0, \epsilon)\} \end{cases}$$

Without loss of generality let us assume that $\epsilon < 1/2$. Then, since $|\tilde{z}_0| = 1$,

$$(5.22) \quad g_\nu(\tilde{z}) < |\tilde{z}|^{-2n+2} < 2^{2n-2}, \quad \tilde{z} \in B(\tilde{z}_0, \epsilon) \cap \tilde{D}_\nu.$$

Now, consider the affine map

$$\tilde{Z} = S\tilde{z} = \frac{\tilde{z} - \tilde{z}_0}{\epsilon}$$

and set

$$\Omega = S(B(\tilde{z}_0, \epsilon) \cap \tilde{H}), \quad \Omega_\nu = S(B(\tilde{z}_0, \epsilon) \cap \tilde{D}_\nu).$$

Define

$$(5.23) \quad h(\tilde{Z}) = 2^{-2n+2} g(\tilde{z}), \quad \tilde{Z} \in \Omega$$

and

$$(5.24) \quad h_\nu(\tilde{Z}) = 2^{-2n+2} g_\nu(\tilde{z}), \quad \tilde{Z} \in \Omega_\nu.$$

Then h_ν is a positive harmonic function on Ω_ν and satisfies $h_\nu = 0$ on $B(0, 1) \cap \partial \Omega_\nu$. Moreover, by (5.22),

$$0 < h_\nu(\tilde{Z}) < 1, \quad \tilde{Z} \in \Omega_\nu.$$

By passing to a subsequence if necessary, it follows from Harnack's principle that $\{h_\nu\}$ converges uniformly on compact subsets of Ω to a positive harmonic function h which satisfies $h = 0$ on $B(0, 1) \cap \partial \Omega$. In view of (5.21), the sequence $\{h_\nu\}$ satisfies the hypothesis of [7, proposition 5.1] and hence from (5.20) and (5.24),

$$\begin{aligned} \lim_{\nu \rightarrow \infty} \left| \frac{\partial^2 G}{\partial x_i \partial x_j}(z_\nu, p_\nu) \right| \left| \partial_z G(z_\nu, p_\nu) \right|^{-1} |z_\nu - p_\nu| \\ = \epsilon \lim_{\nu \rightarrow \infty} \left| \frac{\partial^2 h_\nu}{\partial \tilde{X}_i \partial \tilde{X}_j}(\tilde{Z}_\nu) \right| \left| \partial_{\tilde{Z}} h_\nu(\tilde{Z}_\nu) \right|^{-1} |\tilde{z}_\nu| = \epsilon \left| \frac{\partial^2 h}{\partial \tilde{X}_i \partial \tilde{X}_j}(0) \right| \left| \partial_{\tilde{Z}} h(0) \right|^{-1} \end{aligned}$$

where $\tilde{Z}_\nu = S\tilde{z}_\nu$. Now by the reflection principle, h extends as a harmonic function to a neighbourhood of 0 and hence the quantity on the extreme right of the above equation is finite. This contradicts (5.15) and hence $0 \notin \partial \tilde{H}$.

By Case I and Case II, $0 \notin \overline{\tilde{\mathcal{H}}}$ which contradicts (5.19). Therefore (5.13) holds and the lemma is proved. \square

Recall that if $r > 0$ and I are as in lemma 2.4, then the function $F^\nu(w)$ is defined and smooth on the collar $\mathcal{E}^\nu(r)$.

Proposition 5.5. *There exists $0 < r < 1$, a constant $C > 0$ and an integer I such that*

- (1) $|F^\nu(w)| < C(1 + |w|^{-1})|w|^{-2n+3}$,
- (2) $|(\partial F^\nu / \partial y_i)(w)| < C(1 + |w|^{-1})|w|^{-2n+2}$,
- (3) $|(\partial^2 F^\nu / \partial y_i \partial y_j)(w)| < C(1 + |w|^{-1})|w|^{-2n+1}$

for all $\nu \geq I$ and $w \in \mathcal{E}^\nu(r)$.

Proof. Choose $m > 0$, $0 < r < 1$ and I as in lemma 2.4. Choose $M > 0$ as in lemma 2.5. Modify I and choose a constant C so that proposition 5.3 holds. Modify r and I so that lemma 5.4 holds. Now fix $\nu \geq I$.

- (1) Let $w \in \mathcal{E}^\nu(r)$, $|w| > 1$. Then by lemma 2.4, lemma 2.5 and proposition 5.3,

$$|F^\nu(w)| = \frac{\left| \frac{\partial f_\nu}{\partial p_\gamma}(p_\nu, w) \right|}{\left| \partial_w f_\nu(p_\nu, w) \right|} |\partial_w g^\nu(w)| \leq \frac{M(1 + |w|^{-1})|w|^2}{m} C |w|^{-2n+1} = C_2(1 + |w|^{-1})|w|^{-2n+3}.$$

where $C_1 = MC/m$ is independent of ν and w .

(2) Differentiating $F^\nu(w)$ with respect to y_i ,

$$(5.25) \quad \frac{\partial F^\nu}{\partial y_i} = \frac{-\frac{\partial^2 f_\nu}{\partial p_\gamma \partial y_i}}{|\partial_w f_\nu|} |\partial_w g^\nu| + \frac{1}{4} \frac{\partial f_\nu}{\partial p_\gamma} \frac{\sum_{k=1}^{2n} \frac{\partial f_\nu}{\partial y_k} \frac{\partial^2 f_\nu}{\partial y_k \partial y_i}}{|\partial_w f_\nu|^3} |\partial_w g^\nu| - \frac{1}{4} \frac{\partial f_\nu}{\partial p_\gamma} \frac{1}{|\partial_w f_\nu|} \frac{\sum_{k=1}^{2n} \frac{\partial g_\nu}{\partial y_k} \frac{\partial^2 g_\nu}{\partial y_k \partial y_i}}{|\partial_w g^\nu|}$$

Thus for $w \in \mathcal{E}^\nu(r)$ with $|w| > 1$, by lemma 2.4, lemma 2.5 and proposition 5.3, and the fact that

$$\frac{\frac{\partial g^\nu}{\partial y_k}}{|\partial_w g^\nu|} \leq 2,$$

we have

$$\begin{aligned} \left| \frac{\partial F^\nu}{\partial y_i}(w) \right| &\leq \frac{M(1+|w|^{-1})|w|}{m} C|w|^{-2n+1} + \frac{1}{4} M(1+|w|^{-1})|w|^2 \frac{2nMM|w|^{-1}}{m^3} C|w|^{-2n+1} \\ &\quad + \frac{1}{4} M(1+|w|^{-1})|w|^2 \frac{1}{m} 2n2C|w|^{-2n} \leq C_2(1+|w|^{-1})|w|^{-2n+2}. \end{aligned}$$

(3) In order to prove this estimate, we differentiate (5.25) with respect to y_j and estimate as before. All terms except those of the form

$$\frac{\frac{\partial f_\nu}{\partial p_\gamma}}{|\partial_w f_\nu|} \frac{\frac{\partial^2 g_\nu}{\partial y_k \partial y_i} \frac{\partial^2 g_\nu}{\partial y_l \partial y_j}}{|\partial_w g_\nu|} \quad \text{or} \quad \frac{\frac{\partial f_\nu}{\partial p_\nu}}{|\partial_w f_\nu|} \frac{\frac{\partial g_\nu}{\partial y_k} \frac{\partial g_\nu}{\partial y_l} \frac{\partial^2 g_\nu}{\partial y_k \partial y_i} \frac{\partial^2 g_\nu}{\partial y_l \partial y_j}}{|\partial_w g_\nu|^3}$$

can be estimated from the above by $\text{const.}(1+|w|^{-1})|w|^{-2n+1}$ for $w \in \mathcal{E}^\nu(r)$. Also by lemma 5.4, the above terms can be esitimated from the above by $\text{const.}(1+|w|^{-1})|w|^{-2n+1}$ for $w \in \mathcal{E}^\nu(r_0)$. \square

We now modify the Steps 4 and 5 of chapter 5 [7] to find an upper bound for $\frac{\partial^2 g_\nu}{\partial \bar{w}_\alpha \partial p_\gamma}(p_\nu, w)$.

Proposition 5.6. *There exist $0 < r < 1$ and an integer I such that for $\nu \geq I$ and $w_0 \in \partial D^\nu$, we can find a function $F^*(w)$ (depending on the parameters ν and w_0) of class C^2 on*

$$E = \{w \in D^\nu : |w - w_0| < r|w_0|\}$$

such that

$$H_E F^*(w) = \frac{\partial g_\nu}{\partial p_\gamma}(p_\nu, w), \quad w \in E.$$

Moreover, there exists a constant $C > 0$ independent of ν and $w_0 \in \partial D^\nu$ such that

- (1) $|F^*(w)| < C(1+|w_0|^{-1})|w_0|^{-2n+3}$ in E .
- (2) $|(\partial F^*/\partial y_i)(w_0)| < C(1+|w_0|^{-1})|w_0|^{-2n+2}$, $i = 1, \dots, n$.
- (3) $|\Delta_w F^*(w)| < C(1+|w_0|^{-1})|w_0|^{-2n+1}$ in E .

Proof. Choose $0 < r < 1$, a constant C and an integer I as in proposition 5.5. Now fix $\nu \geq I$ and $w_0 \in \partial D^\nu$ and let

$$B = \{w : |w - w_0| < r|w_0|\}.$$

Then $E = B \cap D^\nu$. Since

$$\frac{\partial g_\nu}{\partial p_\gamma}(p_\nu, w) = H_{D^\nu} F^\nu(w)$$

on D^ν , the function $(\partial g_\nu/\partial p_\gamma)(p_\nu, w)$ is harmonic on E with boundary values

$$(5.26) \quad \begin{cases} F^\nu & ; \quad \text{if } w \in B \cap \partial D^\nu, \\ H_{D^\nu} F^\nu & ; \quad \text{if } w \in \partial B \cap D^\nu. \end{cases}$$

Let u be the harmonic function on E with boundary values

$$u(w) = \begin{cases} 0 & ; \quad \text{if } w \in B \cap \partial D^\nu, \\ H_{D^\nu} F^\nu - F^\nu & ; \quad \text{if } w \in \partial B \cap D^\nu \end{cases}$$

and set

$$F^*(w) = F^\nu(w) + u(w), \quad w \in E.$$

Then

$$H_E F^* = H_E F^\nu + u$$

is a harmonic function on E with boundary values (5.26) and hence

$$H_E F^* = \frac{\partial g_\nu}{\partial p_\gamma}(p_\nu, w)$$

on E and this proves the first part of the proposition.

To prove the second part, note that by proposition 5.5 and continuity of the function

$$|F^\nu(w)|(1+|w|^{-1})^{-1}|w|^{2n-3}$$

up to $\bar{\mathcal{E}}^\nu(r)$, we have

$$|F^\nu(w)| \leq C(1+|w|^{-1})|w|^{-2n+3}, \quad w \in \bar{\mathcal{E}}^\nu(r).$$

In particular, the above holds for $w \in \bar{E}$. Also, since $(1+|w|^{-1})|w|^{-2n+3}$ is superharmonic on \mathbf{C}^n , this also implies that

$$(5.27) \quad |H_{D^\nu} F^\nu(w)| \leq C(1+|w|^{-1})|w|^{-2n+3}, \quad w \in \bar{D}^\nu.$$

Therefore,

$$|H_{D^\nu} F^\nu(w) - F^\nu(w)| \leq 2C(1+|w|^{-1})|w|^{-2n+3}, \quad w \in \partial B \cap D^\nu$$

which implies that

$$|u(w)| \leq 2C(1+|w|^{-1})|w|^{-2n+3}, \quad w \in \bar{E}.$$

Since $E \subset \{w : |w - w_0| < r|w_0|\}$,

$$|F^*(w)| \leq |F_\nu(w)| + |u(w)| \leq 3C(1+|w|^{-1})|w|^{-2n+3} \leq 3C(1-r)^{-2n+2}(1+|w_0|^{-1})|w_0|^{-2n+3}, \quad w \in E$$

which proves (1).

To prove (2), note that from the above calculation

$$|u(w)| \leq 2C(1-r)^{-2n+2}(1+|w_0|^{-1})|w_0|^{-2n+3}, \quad w \in E.$$

Also $u(w) = 0$ for $w \in B \cap \partial D^\nu$. Moreover by lemma 4.1, we can modify the integer I if necessary, to find a $\rho > 0$ which is independent of ν and w_0 such that there exists a ball of radius $\rho|w_0|$ which is externally tangent to ∂D^ν at w_0 . Hence taking $R = \min(\rho|w_0|, r|w_0|)$ in Step 2 of chapter 4 [7], we can find a constant c independent of D^ν and u such that

$$|\partial_w u(w_0)| < \frac{2cC(1-r)^{-2n+2}(1+|w_0|^{-1})|w_0|^{-2n+3}}{\min(r|w_0|, \rho|w_0|)} = \tilde{C}(1+|w_0|^{-1})|w_0|^{-2n+2}$$

where \tilde{C} is independent of ν and $w_0 \in \partial D^\nu$. This together with proposition 5.5 implies

$$\left| \frac{\partial F^*}{\partial y_i}(w_0) \right| \leq \left| \frac{\partial F_\nu}{\partial y_i}(w_0) \right| + \left| \frac{\partial u}{\partial y_i}(w_0) \right| \leq (C + \tilde{C})(1+|w_0|^{-1})|w_0|^{-2n+2}$$

which proves (2).

Finally using the fact that u is harmonic we obtain from proposition 5.5 that

$$|\Delta_w F^*(w)| = |\Delta_w F^\nu(w)| \leq nC(1+|w|^{-1})|w|^{-2n+1} \leq nC(1-r)^{-2n}(1+|w_0|^{-1})|w_0|^{-2n+1}, \quad w \in E$$

and this proves (3). \square

Proposition 5.7. *There exist a constant $C > 0$ and an integer I such that*

$$(5.28) \quad \left| \frac{\partial^2 g_\nu}{\partial \bar{w}_\alpha \partial p_\gamma}(p_\nu, w) \right| < C(1+|w|^{-1})|w|^{-2n+2}$$

for all $\nu \geq I$ and $w \in \bar{D}^\nu$.

Proof. 5.7. Let $0 < r < 1$, $C > 0$ and I be as in proposition 5.6 and fix $\nu \geq I$. By the maximum principle, it suffices to prove (5.28) for $w_0 \in \partial D^\nu$. Given such w_0 , we let F^* be a C^2 -smooth function on

$$E = \{w \in D^\nu : |w - w_0| < r|w_0|\}$$

satisfying the estimates of proposition 5.6. Now consider the affine map

$$W = S(w) = \frac{w - w_0}{r|w_0|}$$

and let $\Omega = S(E)$. Define the functions u and h on Ω by setting

$$u(W) = \frac{\partial g_\nu}{\partial p_\gamma}(p_\nu, w) \quad \text{and} \quad h(W) = F^*(w).$$

Then $u = H_\Omega h$ on Ω and by proposition 5.6,

$$(1) |h(W)| < C(1 + |w_0|^{-1})|w_0|^{-2n+3} \quad \text{in } \Omega,$$

$$(2) \left| \frac{\partial h}{\partial Y_i}(0) \right| = \left| \frac{\partial F^*}{\partial y_i}(w_0) \right| r |w_0| < Cr(1 + |w_0|^{-1})|w_0|^{-2n+3}, \text{ and}$$

$$(3) |\Delta_W h(W)| = |\Delta_w F^*(w)| r^2 |w_0|^2 \leq Cr^2(1 + |w_0|^{-1})|w_0|^{-2n+3} \leq Cr(1 + |w_0|^{-1})|w_0|^{-2n+3} \quad \text{in } \Omega.$$

By lemma 4.1, we can modify the integer I to find a $\rho > 0$ which is independent of ν and w_0 such that there exists a ball B of radius $\rho|w_0|$ which is externally tangent to ∂D^ν at w_0 . Setting $T(B) = \tilde{B}$, we see that the ball $\tilde{B} \subset \mathbb{C}^n \setminus \Omega$ has radius ρ/r and is tangent to $\partial\Omega$ at 0. Let \tilde{B}_2 be the ball with centre same as \tilde{B} and radius $\rho/r + 2$. Hence by [7, pp 60, lemma 5.1',], there exists a constant M depending only on ρ/r such that

$$|\partial_{\overline{W}} u(0)| \leq MC(1 + |w_0|^{-1})|w_0|^{-2n+3}$$

Since

$$\frac{\partial u}{\partial \overline{W}_\alpha}(0) = \frac{\partial^2 g_\nu}{\partial \overline{w}_\alpha \partial p_\gamma}(p, w_0) r |w_0|,$$

we have

$$\left| \frac{\partial^2 g_\nu}{\partial \overline{w}_\alpha \partial p_\gamma}(p_\nu, w_0) \right| \leq \frac{MC}{r}(1 + |w_0|^{-1})|w_0|^{-2n+2}$$

which proves the proposition. \square

Proposition 5.8. *Let $w^\nu \in \partial D^\nu$ be such that $\{w^\nu\}$ converges to $w^0 \in \partial\mathcal{H} = \partial D(p_0)$. Then*

$$\lim_{\nu \rightarrow \infty} \frac{\partial^2 g_\nu}{\partial \overline{w}_\alpha \partial p_\gamma}(p_\nu, w^\nu) = \frac{\partial^2 g}{\partial \overline{w}_\alpha \partial p_\gamma}(p_0, w^0).$$

Proof. This follows from standard boundary elliptic regularity arguments and the fact that D^ν is C^∞ -close to D . \square

Proposition 5.9. $\lim_{\nu \rightarrow \infty} \frac{\partial^2 \lambda_\nu}{\partial p_\gamma \partial \overline{p}_\gamma}(p_\nu) = \frac{\partial^2 \lambda}{\partial p_\gamma \partial \overline{p}_\gamma}(p_0).$

Proof. By proposition 2.3 and (5.3), we only need to prove that

$$(5.29) \quad \lim_{\nu \rightarrow \infty} \int_{\partial D^\nu} k_1^{\nu\gamma}(w) \frac{\frac{\partial g^\nu}{\partial \overline{w}_\alpha}(w)}{|\partial_w g^\nu(w)|} \frac{\partial^2 g_\nu}{\partial w_\alpha \partial \overline{p}_\gamma}(p_\nu, w) \frac{\partial g^\nu}{\partial n_w}(w) dS_w \\ = \int_{\partial\mathcal{H}} k_1^\gamma(p_0, w) \frac{\frac{\partial g}{\partial \overline{w}_\alpha}(p_0, w)}{|\partial_w g(p_0, w)|} \frac{\partial^2 g}{\partial w_\alpha \partial \overline{p}_\gamma}(p_0, w) \frac{\partial g}{\partial n_w}(p_0, w) dS_w.$$

Let $R > 1$. Then by the arguments of the proof of proposition 4.4 together with proposition 5.8, we have

$$(5.30) \quad \lim_{\nu \rightarrow \infty} \int_{B(0, R) \cap \partial D^\nu} k_1^{\nu\gamma}(w) \frac{\frac{\partial g^\nu}{\partial \overline{w}_\alpha}(w)}{|\partial_w g^\nu(w)|} \frac{\partial^2 g_\nu}{\partial w_\alpha \partial \overline{p}_\gamma}(p_\nu, w) \frac{\partial g^\nu}{\partial n_w}(w) dS_w \\ = \int_{B(0, R) \cap \partial\mathcal{H}} k_1^\gamma(p_0, w) \frac{\frac{\partial g}{\partial \overline{w}_\alpha}(p_0, w)}{|\partial_w g(p_0, w)|} \frac{\partial^2 g}{\partial w_\alpha \partial \overline{p}_\gamma}(p_0, w) \frac{\partial g}{\partial n_w}(p_0, w) dS_w.$$

To estimate the above integrals outside $B(0, R)$, note that by corollary 2.6, there exist a constant C and an integer I such that

$$|k_1^{\nu\gamma}(w)| \leq C|w|^2, \quad w \in \partial D^\nu, |w| > 1$$

for $\nu \geq I$. In view of proposition 5.7, we can modify C and I so that

$$\left| \frac{\partial^2 g_\nu}{\partial \overline{w}_\alpha \partial p_\gamma}(p_\nu, w) \right| \leq C|w|^{-2n+2}, \quad w \in \partial D^\nu, |w| > 1$$

for $\nu \geq I$. Therefore,

$$(5.31) \quad \left| \int_{B^c(0, R) \cap \partial D^\nu} k_1^\nu(w) \frac{\partial^2 g_\nu}{\partial w_\alpha \partial \overline{p}_\gamma}(p_\nu, w) \frac{\frac{\partial g^\nu}{\partial \overline{w}_\alpha}(w)}{|\partial_w g^\nu(w)|} \frac{\partial g^\nu}{\partial n_w}(w) dS_w \right| \\ \leq C^2 R^{-2n+4} \int_{B^c(0, R) \cap \partial D^\nu} \left(-\frac{\partial g^\nu}{\partial n_w}(w) \right) dS_w.$$

for $\nu \geq I$. Again

$$\int_{B^c(0,R) \cap \partial D^\nu} \left(-\frac{\partial g^\nu}{\partial n_w}(w) \right) dS_w \leq \int_{\partial D^\nu} \left(-\frac{\partial g^\nu}{\partial n_w}(w) \right) dS_w = 2(n-1)\sigma_{2n}$$

and hence from (5.31)

$$(5.32) \quad \left| \int_{B^c(0,R) \cap \partial D^\nu} k_1^\nu(w) \frac{\partial^2 g_\nu}{\partial w_\alpha \partial \bar{p}_\gamma}(p_\nu, w) \frac{\frac{\partial g^\nu}{\partial \bar{w}_\alpha}(w)}{|\partial_w g^\nu(w)|} \frac{\partial g^\nu}{\partial n_w}(w) dS_w \right| = O(R^{-2n+4})$$

uniformly for all $\nu \geq I$. Also by (2.7), we can modify the above constant C so that

$$|k_1^\gamma(p_0, w)| \leq C|w|^2 \quad \text{and} \quad \left| \frac{\partial^2 g}{\partial \bar{w}_\alpha \partial p_\gamma}(p_0, w) \right| \leq C|w|^{-2n+2}$$

for $w \in \partial \mathcal{H}$ with $|w| > 1$. As above we obtain

$$(5.33) \quad \left| \int_{B^c(0,R) \cap \partial \mathcal{H}} k_1^\gamma(p_0, w) \frac{\frac{\partial g}{\partial \bar{w}_\alpha}(p_0, w)}{|\partial_w g(p_0, w)|} \frac{\partial^2 g}{\partial w_\alpha \partial \bar{p}_\gamma}(p_0, w) \frac{\partial g}{\partial n_w}(p_0, w) dS_w \right| = O(R^{-2n+4}).$$

From (5.30), (5.32) and (5.33) it follows that (5.29) holds. \square

Proof of Theorem 1.3. In view of proposition 4.4, we only need to prove that

$$\lim_{\nu \rightarrow \infty} \frac{\partial^2 \lambda_\nu}{\partial p_\alpha \partial \bar{p}_\beta}(p_\nu) = \frac{\partial^2 \lambda}{\partial p_\alpha \partial \bar{p}_\beta}(p_0).$$

But this follows from proposition 5.9 by a unitary change of coordinates. \square

6. HOLOMORPHIC SECTIONAL CURVATURE

In this section we prove theorem 1.1 by deriving the asymptotics of the terms in (1.1).

Lemma 6.1. *We have*

- (1) $\lim_{\nu \rightarrow \infty} (g_\nu)_{\alpha \bar{\beta}}(p_\nu) (\psi_\nu(p_\nu))^2 = (2n-2)\psi_\alpha(0)\psi_{\bar{\beta}}(0),$
- (2) $\lim_{\nu \rightarrow \infty} \frac{\partial (g_\nu)_{\alpha \bar{\beta}}}{\partial z_\gamma}(p_\nu) (\psi_\nu(p_\nu))^3 = -2(2n-2)\psi_\alpha(0)\psi_{\bar{\beta}}(0)\psi_\gamma(0),$
- (3) $\lim_{\nu \rightarrow \infty} \frac{\partial^2 (g_\nu)_{\alpha \bar{\beta}}}{\partial z_\gamma \partial \bar{z}_\delta}(p_\nu) (\psi_\nu(p_\nu))^4 = 6(2n-2)\psi_\alpha(0)\psi_{\bar{\beta}}(0)\psi_\delta(0).$

Proof. Let \mathcal{H} be the half space

$$\mathcal{H} = \left\{ z \in \mathbf{C}^n : 2\Re\left(\sum_{i=1}^n \psi_i(0)z_i\right) - 1 < 0 \right\} = \{z \in \mathbf{C}^n : 2\Re z_n - 1 < 0\}.$$

From [1, (1.4)], the Robin function for \mathcal{H} is given by

$$\Lambda_{\mathcal{H}}(z) = -\left(\frac{|\partial \psi(0)|}{2\Re\left(\sum_{i=1}^n \psi_i(0)z_i\right) - 1} \right)^{2n-2} = -\left(2\Re\left(\sum_{i=1}^n \psi_i(0)z_i\right) - 1 \right)^{-2n+2}$$

so that

- $\Lambda_{\mathcal{H}}(0) = -1,$
- $(\Lambda_{\mathcal{H}})_a(0) = -(2n-2)\psi_a(0),$
- $(\Lambda_{\mathcal{H}})_{ab}(0) = -(2n-2)(2n-1)\psi_a(0)\psi_b(0),$
- $(\Lambda_{\mathcal{H}})_{abc}(0) = -(2n-2)(2n-1)(2n)\psi_a(0)\psi_b(0)\psi_c(0)$ and
- $(\Lambda_{\mathcal{H}})_{abcd}(0) = -(2n-2)(2n-1)(2n)(2n+1)\psi_a(0)\psi_b(0)\psi_c(0)\psi_d(0)$

where the indices a, b, c, d refer to either holomorphic or conjugate holomorphic derivatives. Hence by theorem 1.2, we get

- $\Lambda_\nu(p_\nu) (\psi_\nu(p_\nu))^{2n-2} \rightarrow -1,$
- $\Lambda_{\nu a}(p_\nu) (\psi_\nu(p_\nu))^{2n-1} \rightarrow (2n-2)\psi_a(0),$
- $\Lambda_{\nu ab}(p_\nu) (\psi_\nu(p_\nu))^{2n} \rightarrow -(2n-2)(2n-1)\psi_a(0)\psi_b(0),$
- $\Lambda_{\nu abc}(p_\nu) (\psi_\nu(p_\nu))^{2n+1} \rightarrow -(2n-2)(2n-1)(2n)\psi_a(0)\psi_b(0)\psi_c(0)$ and
- $\Lambda_{\nu abcd}(p_\nu) (\psi_\nu(p_\nu))^{2n+2} \rightarrow -(2n-2)(2n-1)(2n)(2n+1)\psi_a(0)\psi_b(0)\psi_c(0)\psi_d(0).$

Now

$$(6.1) \quad g_{\alpha\bar{\beta}} = \frac{\partial^2 \log(-\Lambda)}{\partial z_\alpha \partial \bar{z}_\beta} = \frac{\Lambda_{\alpha\bar{\beta}}}{\Lambda} - \frac{\Lambda_\alpha \Lambda_{\bar{\beta}}}{\Lambda^2}.$$

Multiplying both sides of this equation by ψ^2 , we get

$$g_{\alpha\bar{\beta}} \psi^2 = \frac{\Lambda_{\alpha\bar{\beta}} \psi^{2n}}{\Lambda \psi^{2n-2}} - \frac{(\Lambda_\alpha \psi^{2n-1})(\Lambda_{\bar{\beta}} \psi^{2n-1})}{(\Lambda \psi^{2n-2})^2}.$$

It follows that

$$\lim_{\nu \rightarrow \infty} g_{\nu\alpha\bar{\beta}}(p_\nu) (\psi_\nu(p_\nu))^2 = (2n-2)\psi_\alpha(0)\psi_{\bar{\beta}}(0)$$

which is (i).

Differentiating (6.1) with respect to z_γ , we obtain

$$(6.2) \quad \frac{\partial g_{\alpha\bar{\beta}}}{\partial z_\gamma} = \frac{\Lambda_{\alpha\bar{\beta}\gamma}}{\Lambda} - \left(\frac{\Lambda_{\alpha\bar{\beta}} \Lambda_\gamma}{\Lambda^2} + \frac{\Lambda_{\alpha\gamma} \Lambda_{\bar{\beta}}}{\Lambda^2} + \frac{\Lambda_{\bar{\beta}\gamma} \Lambda_\alpha}{\Lambda^2} \right) + \frac{2\Lambda_\alpha \Lambda_{\bar{\beta}} \Lambda_\gamma}{\Lambda^3}.$$

Multiplying both sides of this equation by ψ^3 , we get

$$\begin{aligned} \frac{\partial g_{\alpha\bar{\beta}}}{\partial z_\gamma} \psi^3 &= \frac{\Lambda_{\alpha\bar{\beta}\gamma} \psi^{2n+1}}{\Lambda \psi^{2n-2}} - \left(\frac{(\Lambda_{\alpha\bar{\beta}} \psi^{2n})(\Lambda_\gamma \psi^{2n-1})}{(\Lambda \psi^{2n-2})^2} + \frac{(\Lambda_{\alpha\gamma} \psi^{2n})(\Lambda_{\bar{\beta}} \psi^{2n-1})}{(\Lambda \psi^{2n-2})^2} + \frac{(\Lambda_{\bar{\beta}\gamma} \psi^{2n})(\Lambda_\alpha \psi^{2n-1})}{(\Lambda \psi^{2n-2})^2} \right) \\ &\quad + \frac{2(\Lambda_\alpha \psi^{2n-1})(\Lambda_{\bar{\beta}} \psi^{2n-1})(\Lambda_\gamma \psi^{2n-1})}{(\Lambda \psi^{2n-2})^3}. \end{aligned}$$

It follows that

$$\lim_{\nu \rightarrow \infty} \frac{\partial g_{\nu\alpha\bar{\beta}}}{\partial z_\gamma}(p_\nu) \psi_\nu(p_\nu)^3 = -2(2n-2)\psi_\alpha(0)\psi_{\bar{\beta}}(p)\psi_\gamma(0)$$

which is (ii).

Differentiating (6.2) with respect to \bar{z}_δ , we obtain

$$\begin{aligned} \frac{\partial^2 g_{\alpha\bar{\beta}}}{\partial z_\gamma \partial \bar{z}_\delta} &= \frac{\Lambda_{\alpha\bar{\beta}\gamma\bar{\delta}}}{\Lambda} - \left(\frac{\Lambda_{\alpha\bar{\beta}\gamma} \Lambda_{\bar{\delta}}}{\Lambda^2} + \frac{\Lambda_{\alpha\bar{\beta}\delta} \Lambda_\gamma}{\Lambda^2} + \frac{\Lambda_{\alpha\gamma\bar{\delta}} \Lambda_{\bar{\beta}}}{\Lambda^2} + \frac{\Lambda_{\bar{\beta}\gamma\bar{\delta}} \Lambda_\alpha}{\Lambda^2} \right) - \left(\frac{\Lambda_{\alpha\bar{\beta}} \Lambda_{\gamma\bar{\delta}}}{\Lambda^2} + \frac{\Lambda_{\alpha\gamma} \Lambda_{\bar{\beta}\bar{\delta}}}{\Lambda^2} + \frac{\Lambda_{\alpha\bar{\delta}} \Lambda_{\bar{\beta}\gamma}}{\Lambda^2} \right) \\ &\quad + 2 \left(\frac{\Lambda_{\alpha\bar{\beta}} \Lambda_\gamma \Lambda_{\bar{\delta}}}{\Lambda^3} + \frac{\Lambda_{\alpha\gamma} \Lambda_{\bar{\beta}} \Lambda_{\bar{\delta}}}{\Lambda^3} + \frac{\Lambda_{\bar{\beta}\gamma} \Lambda_\alpha \Lambda_{\bar{\delta}}}{\Lambda^3} + \frac{\Lambda_{\alpha\bar{\delta}} \Lambda_{\bar{\beta}} \Lambda_\gamma}{\Lambda^3} + \frac{\Lambda_{\bar{\beta}\bar{\delta}} \Lambda_\alpha \Lambda_\gamma}{\Lambda^3} + \frac{\Lambda_{\gamma\bar{\delta}} \Lambda_\alpha \Lambda_{\bar{\beta}}}{\Lambda^3} \right) - \frac{6\Lambda_\alpha \Lambda_{\bar{\beta}} \Lambda_\gamma \Lambda_{\bar{\delta}}}{\Lambda^4}. \end{aligned}$$

Multiplying both sides by ψ^4 , this equation can be written in a form where Λ is multiplied by ψ^{2n-2} and first, second, third and fourth order derivatives of Λ are multiplied by ψ^{2n-1} , ψ^{2n} , ψ^{2n+1} and ψ^{2n+2} respectively. It follows that

$$\lim_{\nu \rightarrow \infty} \frac{\partial^2 g_{\nu\alpha\bar{\beta}}}{\partial z_\gamma \partial \bar{z}_\delta}(p_\nu) (\psi_\nu(p_\nu))^4 = 6(2n-2)\psi_\alpha(0)\psi_{\bar{\beta}}(0)\psi_\gamma(0)\psi_{\bar{\delta}}(0)$$

which is (iii). □

To obtain finer asymptotics of the derivatives of Λ_ν along $\{p_\nu\}$, we need the following:

Lemma 6.2. *Let $1 \leq \alpha \leq n-1$. Then*

$$\lim_{\nu \rightarrow \infty} \frac{(\psi_\nu)_\alpha(p_\nu)}{\psi_\nu(p_\nu)} = \frac{1}{2}(\psi_{\alpha n}(0) + \psi_{\alpha \bar{n}}(0)).$$

Proof. Fix a ν and define the function f on $[0, 1]$ by

$$(6.3) \quad f(t) = \psi_\nu(tp_\nu) = \psi_\nu(0, \dots, 0, -\delta_\nu t).$$

By Taylor's theorem

$$f(1) = f(0) + f'(0) + \frac{1}{2}f''(s)$$

for some $s \in (0, 1)$. Therefore, by successive application of the Chain rule to (6.3), we obtain

$$(6.4) \quad \psi_\nu(p_\nu) = -\delta_\nu((\psi_\nu)_n(0) + (\psi_\nu)_{\bar{n}}(0)) + \frac{\delta_\nu^2}{2}((\psi_\nu)_{nn}(\zeta_\nu) + 2(\psi_\nu)_{n\bar{n}}(\zeta_\nu) + (\psi_\nu)_{\bar{n}\bar{n}}(\zeta_\nu))$$

where $\zeta_\nu = sp_\nu$.

Now fix $1 \leq \alpha \leq n-1$ and define the function g on $[0, 1]$ by

$$(6.5) \quad g(t) = (\psi_\nu)_\alpha(tp_\nu) = (\psi_\nu)_\alpha(0, \dots, 0, -\delta_\nu t).$$

By Taylor's theorem

$$g(1) = g(0) + g'(0) + \frac{1}{2}g''(s)$$

for some $s' \in (0, 1)$. Therefore, by successive application of the chain rule to (6.5), we obtain

$$(6.6) \quad (\psi_\nu)_\alpha(p_\nu) = -\delta_\nu((\psi_\nu)_{\alpha n}(0) + (\psi_\nu)_{\alpha \bar{n}}(0)) + \frac{\delta_\nu^2}{2}((\psi_\nu)_{\alpha n n}(\eta_\nu) + 2(\psi_\nu)_{\alpha n \bar{n}}(\eta_\nu) + (\psi_\nu)_{\alpha \bar{n} \bar{n}}(\eta_\nu))$$

where $\eta_\nu = s'p_\nu$. It is now evident from (6.4) and (6.6), that

$$\lim_{\nu \rightarrow \infty} \frac{(\psi_\nu)_\alpha(p_\nu)}{\psi_\nu(p_\nu)} = \frac{1}{2}(\psi_{\alpha n}(0) + \psi_{\alpha \bar{n}}(0))$$

and the lemma is proved. \square

Using this lemma and theorem 1.3, we obtain the following finer asymptotics of the first and second order derivatives of Λ_ν along $\{p_\nu\}$.

Lemma 6.3. *Let $1 \leq \alpha \leq n-1$ and $1 \leq \beta \leq n$. Then*

- (i) $\lim_{\nu \rightarrow \infty} \Lambda_{\nu\alpha}(p_\nu)(\psi_\nu(p_\nu))^{2n-2} = \lambda_\alpha(0) + (2n-2)C_\alpha$,
- (ii) $\lim_{\nu \rightarrow \infty} \Lambda_{\alpha\bar{\beta}}(p_\nu)(\psi_\nu(p_\nu))^{2n-1} = -(2n-2)\lambda_\alpha(0)\psi_{\bar{\beta}}(0) - (2n-2)(2n-1)\psi_{\bar{\beta}}(0)C_\alpha + (2n-2)\psi_{\alpha\bar{\beta}}(0)$

where $C_\alpha = \frac{1}{2}(\psi_{\alpha n}(0) + \psi_{\alpha \bar{n}}(0))$.

Proof. The normalised robin function

$$(6.7) \quad \lambda(z) = \begin{cases} \Lambda(z)(\psi(z))^{2n-2} & \text{if } z \in D \\ -|\partial\psi(z)|^{2n-2} & \text{if } z \in \partial D \end{cases}$$

associated to (D, ψ) is C^2 on \overline{D} . In particular, $\lambda(0) = -1$. Differentiating λ with respect to z_α , we obtain

$$\Lambda_\alpha \psi^{2n-2} = \lambda_\alpha - (2n-2)\lambda\psi^{-1}\psi_\alpha.$$

Hence by theorems 1.2, 1.3 and lemma 6.2,

$$\lim_{\nu \rightarrow \infty} \Lambda_{\nu\alpha}(p_\nu)(\psi_\nu(p_\nu))^{2n-2} = \lambda_\alpha(0) + (2n-2)C_\alpha$$

which is (i). Similarly differentiating (6.7) with respect to z_α followed by \bar{z}_β we obtain

$$\Lambda_{\alpha\bar{\beta}} \psi^{2n-1} = \lambda_{\alpha\bar{\beta}} \psi - (2n-2)(\lambda_\alpha \psi_{\bar{\beta}} + \lambda_{\bar{\beta}} \psi_\alpha) + (2n-2)(2n-1)\lambda\psi^{-1}\psi_\alpha \psi_{\bar{\beta}} - (2n-2)\lambda\psi_{\alpha\bar{\beta}}$$

Again by theorems 1.2, 1.3 and lemma 6.2,

$$\lim_{\nu \rightarrow \infty} \Lambda_{\alpha\bar{\beta}}(p_\nu)(\psi_\nu(p_\nu))^{2n-1} = -(2n-2)\lambda_\alpha(0)\psi_{\bar{\beta}}(0) - (2n-2)(2n-1)\psi_{\bar{\beta}}(0)C_\alpha + (2n-2)\psi_{\alpha\bar{\beta}}(0)$$

which is (ii). \square

Lemma 6.4. *Let $1 \leq \alpha \leq n-1$ and $1 \leq \beta \leq n$. Then*

$$\lim_{\nu \rightarrow \infty} g_{\nu\alpha\bar{\beta}}(p_\nu)(\psi_\nu(p_\nu)) = (2n-2) \left(\frac{1}{2} \{ \psi_{\alpha n}(0) + \psi_{\alpha \bar{n}}(0) \} \psi_{\bar{\beta}}(0) - \psi_{\alpha\bar{\beta}}(0) \right).$$

Proof. We have

$$g_{\alpha\bar{\beta}} = \frac{\partial^2 \log(-\Lambda)}{\partial z_\alpha \partial \bar{z}_\beta} = \frac{\Lambda_{\alpha\bar{\beta}}}{\Lambda} - \frac{\Lambda_\alpha \Lambda_{\bar{\beta}}}{\Lambda^2}.$$

Multiplying both sides of this equation by ψ , we get

$$(6.8) \quad g_{\alpha\bar{\beta}} \psi = \frac{\Lambda_{\alpha\bar{\beta}} \psi^{2n-1}}{\Lambda \psi^{2n-2}} - \frac{(\Lambda_\alpha \psi^{2n-2})(\Lambda_{\bar{\beta}} \psi^{2n-1})}{(\Lambda \psi^{2n-2})^2}.$$

By the proof of lemma 6.1

$$\Lambda_\nu(p_\nu)(\psi_\nu(p_\nu))^{2n-2} \rightarrow -1$$

and

$$\Lambda_{\nu\bar{\beta}}(p_\nu)(\psi_\nu(p_\nu))^{2n-1} \rightarrow (2n-2)\psi_{\bar{\beta}}(0).$$

Therefore using lemma 6.3 we obtain from (6.8),

$$\begin{aligned} \lim_{\nu \rightarrow \infty} g_{\nu\alpha\bar{\beta}}(p_\nu)\psi_\nu(p_\nu) &= (2n-2)\lambda_\alpha(0)\psi_{\bar{\beta}}(0) + (2n-2)(2n-1)\psi_{\bar{\beta}}(0)C - (2n-2)\psi_{\alpha\bar{\beta}}(0) \\ &\quad - \{ \lambda_\alpha(0) + (2n-2)C_\alpha \} \{ (2n-2)\psi_{\bar{\beta}}(0) \} \end{aligned}$$

Simplifying the right hand side we obtain

$$\begin{aligned} \lim_{\nu \rightarrow \infty} g_{\nu\alpha\bar{\beta}}(p_\nu)\psi_\nu(p_\nu) &= (2n-2)(\psi_{\bar{\beta}}(0)C_\alpha - \psi_{\alpha\bar{\beta}}(0)) \\ &= (2n-2)\left(\frac{1}{2}\{\psi_{\alpha n}(0) + \psi_{\alpha\bar{n}}(0)\}\psi_{\bar{\beta}}(0) - \psi_{\alpha\bar{\beta}}(0)\right). \end{aligned}$$

□

Since we do not have any information about the third order derivatives of $\lambda(p) = \psi^{2n-2}\Lambda(p)$ near the boundary of D , the above method fails to give finer asymptotics of $\Lambda_{\nu\alpha\bar{\beta}\gamma}$. However by proposition 2.1, the function

$$(6.9) \quad g(p, w) = \psi(p)^{2n-2}G(p, z)$$

where $w = (z-p)/(-\psi(p))$, is C^2 up to $\mathcal{D} \cup \partial\mathcal{D}$ and for each $p \in D$, $\frac{\partial g}{\partial p_\alpha}(p)$ and $\frac{\partial^2 g}{\partial p_\alpha \partial \bar{p}_\beta}(p)$ are harmonic functions of $w \in \bar{D}(p)$ and hence can be differentiated infinitely often with respect to w . Moreover

$$(6.10) \quad \frac{\partial g}{\partial p_\alpha}(p, 0) = \frac{\partial \lambda}{\partial p_\alpha}(p) \quad \text{and} \quad \frac{\partial^2 g}{\partial p_\alpha \partial \bar{p}_\beta}(p, 0) = \frac{\partial^2 \lambda}{\partial p_\alpha \partial \bar{p}_\beta}.$$

In the following, we exploit these properties to calculate finer asymptotics of $\Lambda_{\nu\alpha\bar{\beta}\gamma}$ by expressing it in terms of mixed derivatives of g_ν .

By [7, Proposition 6.1], the functions

$$(6.11) \quad \begin{cases} G_\alpha(p, z) &= \left(\frac{\partial G}{\partial p_\alpha} + \frac{\partial G}{\partial z_\alpha}\right)(p, z), \\ G_{\alpha\bar{\beta}}(p, z) &= \left(\frac{\partial G_\alpha}{\partial \bar{p}_\beta} + \frac{\partial G_\alpha}{\partial \bar{z}_\beta}\right)(p, z) \end{cases}$$

are real analytic, symmetric function in $D \times D$ and are harmonic in z and in p . By [7, 6.14]

$$(6.12) \quad \Lambda_{\alpha\bar{\beta}\gamma}(p) = 2\frac{\partial G_{\alpha\bar{\beta}}}{\partial z_\gamma}(p, p)$$

By [7, Proposition 6.2], the functions

$$(6.13) \quad \begin{cases} g_0(p, w) &= g(p, w) + \frac{1}{n-1} \sum_{i=1}^n w_i \frac{\partial g}{\partial w_i}, \\ g_\alpha(p, w) &= \psi(p) \frac{\partial g}{\partial p_\alpha}(p, w) - (n-1)\psi_\alpha(p)(g_0(p, w) + \overline{g_0(p, w)}) \end{cases}$$

are harmonic functions of $w \in D(p)$ for each $p \in \bar{D}$. From [7, page 83],

$$(6.14) \quad \frac{\partial G_{\alpha\bar{\beta}}}{\partial z_\gamma}(p, p) = -(\psi(p))^{-2n-1} \left\{ -2n\psi_{\bar{\beta}}(p) \frac{\partial g_\alpha}{\partial w_\gamma}(p, 0) + \psi(p) \frac{\partial^2 g_\alpha}{\partial w_\gamma \partial \bar{p}_\beta}(p, 0) \right\}$$

Combining (6.12) and (6.14),

$$(6.15) \quad \Lambda_{\alpha\bar{\beta}\gamma}(p)(\psi(p))^{2n} = 4n \frac{\psi_{\bar{\beta}}(p)}{\psi(p)} \frac{\partial g_\alpha}{\partial w_\gamma}(p, 0) - \frac{\partial^2 g_\alpha}{\partial w_\gamma \partial \bar{p}_\beta}(p, 0)$$

Lemma 6.5. *Let $1 \leq \alpha, \gamma \leq n$ and $1 \leq \beta \leq n-1$. Then*

$$\lim_{\nu \rightarrow \infty} \Lambda_{\nu\alpha\bar{\beta}\gamma}(p_\nu)(\psi_\nu(p_\nu))^{2n}$$

exists and is finite.

Proof. By (6.15) and lemma 6.2, we only need to prove that

$$\lim_{\nu \rightarrow \infty} \frac{\partial g_{\nu\alpha}}{\partial w_\gamma}(p_\nu, 0) \quad \text{and} \quad \lim_{\nu \rightarrow \infty} \frac{\partial^2 g_{\nu\alpha}}{\partial w_\gamma \partial \bar{p}_\beta}(p_\nu, 0)$$

exist and are finite.

Now $g_{\nu\alpha}(p_\nu, w)$ is a harmonic function of $w \in D^\nu$. To estimate the boundary values of these functions, note that the first term of $g_{\nu 0}(p_\nu, w)$, i.e., $g_\nu(p_\nu, w)$ is bounded by $|w|^{-2n+2}$ for all ν and by proposition 5.3, the second term is bounded by $C|w|^{-2n+2}$ for all large ν . Therefore, from (6.13)

$$(6.16) \quad |g_{\nu 0}(p_\nu, w)| \leq C|w|^{-2n+2}, \quad w \in \partial D^\nu$$

for all large ν . Again, by proposition 4.3, $|\frac{\partial g_\nu}{\partial p_\alpha}(p_\nu, w)|$ is bounded by $C(1+|w|^{-1})|w|^{-2n+3}$ for all large ν . Also $\psi_\nu(p_\nu)$ and $\psi_{\nu\alpha}(p_\nu)$ are bounded by a constant C for all large ν . Hence from (6.13) and (6.16),

$$(6.17) \quad |g_{\nu\alpha}(p_\nu, w)| \leq C(1+|w|^{-1})|w|^{-2n+3}, \quad w \in \partial D^\nu$$

for all large ν .

Choose $r > 0$ such that $\overline{B}(0, r) \subset \mathcal{H}$. Since D^ν converges in the Hausdorff sense to \mathcal{H} , there exists an integer I such that $\overline{B}(0, r) \subset D^\nu$ for all $\nu \geq I$. Therefore

$$(6.18) \quad |w| > r$$

for all $\nu \geq I$ and $w \in \partial D^\nu$. Hence from (6.17),

$$|g_{\nu\alpha}(p_\nu, w)| \leq Cr^{-2n+3}(1+r^{-1}), \quad w \in \partial D^\nu$$

for all large ν . Therefore, $g_{\nu\alpha}(p_\nu, w)$ is uniformly bounded on $B(0, r)$ for all large ν . Moreover, by [7, Proposition 6.2] and the fact that $\frac{\partial g_\nu}{\partial p_\alpha}(p_\nu, 0) = \frac{\partial \lambda_\nu}{\partial p_\alpha}(p_\nu)$,

$$(6.19) \quad g_{\nu\alpha}(p_\nu, 0) = \psi_\nu(p_\nu) \frac{\partial \lambda_\nu}{\partial p_\alpha}(p_\nu) - (2n-2)\psi_{\nu\alpha}(p_\nu)\lambda(p_\nu)$$

which converges. It follows from Harnack's principle that

$$\lim_{\nu \rightarrow \infty} \frac{\partial g_{\nu\alpha}}{\partial w_\gamma}(p_\nu, 0)$$

exists.

Now differentiating (6.13) with respect to \overline{p}_β , we obtain

$$(6.20) \quad \frac{\partial g_0}{\partial \overline{p}_\beta}(p, w) = \frac{\partial g}{\partial \overline{p}_\beta} + \frac{1}{n-1} \sum_{i=1}^n w_i \frac{\partial^2 g}{\partial \overline{p}_\beta \partial w_i}$$

and

$$(6.21) \quad \begin{aligned} \frac{\partial g_\alpha}{\partial \overline{p}_\beta}(p, w) &= \psi(p) \frac{\partial^2 g}{\partial p_\alpha \partial \overline{p}_\beta}(p, w) + \psi_{\overline{\beta}}(p) \frac{\partial g}{\partial p_\alpha}(p, w) - (n-1)\psi_\alpha(p) \left(\frac{\partial g_0}{\partial \overline{p}_\beta}(p, w) + \overline{\frac{\partial g_0}{\partial p_\beta}(p, w)} \right) \\ &\quad - (n-1)\psi_{\alpha\overline{\beta}}(p) (g_0(p, w) + \overline{g_0(p, w)}) \end{aligned}$$

which are harmonic functions of $w \in D$. As above $|\frac{\partial g_\nu}{\partial \overline{p}_\beta}|$ is bounded by $C(1+|w|^{-1})|w|^{-2n+3}$ for all large ν . Also, by proposition 5.7, $|\frac{\partial^2 g_\nu}{\partial \overline{p}_\beta \partial w_i}|$ is bounded by $C(1+|w|^{-1})|w|^{-2n+2}$ for all large ν . It follows that

$$(6.22) \quad \left| \frac{\partial g_{\nu 0}}{\partial \overline{p}_\beta}(p_\nu, w) \right| \leq C|w|^{-2n+3}, w \in \partial D^\nu$$

for all large ν . From proposition 2.1, for $1 \leq \gamma \leq n$, $p \in D$

$$\left| \frac{\partial^2 g}{\partial p_\gamma \partial \overline{p}_\gamma}(p, w) \right| \leq |k_2^\gamma(p, w)| |\partial_w g(p, w)| + 2|k_1^\gamma| \sum_{i=1}^n \left| \frac{\partial^2 g}{\partial w_i \partial \overline{p}_\gamma} \right|, \quad w \in \partial D(p).$$

It follows that

$$\left| \frac{\partial^2 g_\nu}{\partial p_\gamma \partial \overline{p}_\gamma}(p_\nu, w) \right| \leq C(1+|w|^{-1}+|w|^{-2})|w|^{-2n+4}, w \in \partial D^\nu$$

and hence by a unitary change of coordinates

$$\left| \frac{\partial^2 g_\nu}{\partial p_\alpha \partial \overline{p}_\beta}(p_\nu, w) \right| \leq C(1+|w|^{-1}+|w|^{-2})|w|^{-2n+4}, w \in \partial D^\nu$$

for all large ν . Thus

$$\left| \frac{\partial g_{\nu\alpha}}{\partial \overline{p}_\beta}(p_\nu, w) \right| \leq C(1+|w|^{-1}+|w|^{-2})|w|^{-2n+4} \leq Cr^{-2n+4}(1+r^{-1}+r^{-2}), \quad w \in \partial D^\nu$$

for all large ν . Therefore, the sequence $\{\frac{\partial g_{\nu\alpha}}{\partial \overline{p}_\beta}(p_\nu, w)\}$ is uniformly bounded on $B(0, r)$. Moreover,

$$\frac{\partial g_{\nu\alpha}}{\partial \overline{p}_\beta}(p_\nu, 0) = \psi_\nu(p_\nu) \frac{\partial^2 \lambda_\nu}{\partial p_\alpha \partial \overline{p}_\beta}(p_\nu) + \psi_{\nu\overline{\beta}}(p_\nu) \frac{\partial \lambda_\nu}{\partial p_\alpha}(p_\nu) - (2n-2)\psi_{\nu\alpha}(p_\nu) \frac{\partial \lambda_\nu}{\partial \overline{p}_\beta}(p_\nu) - (2n-2)\psi_{\nu\alpha\overline{\beta}}(p_\nu)\lambda_\nu(p_\nu)$$

which converges. It follows from Harnack's principle that

$$\lim_{\nu \rightarrow \infty} \frac{\partial^2 g_{\nu\alpha}}{\partial w_\gamma \partial \overline{p}_\beta}(p_\nu, 0)$$

exists. □

Lemma 6.6. *Let $1 \leq \alpha, \gamma \leq n$ and $1 \leq \beta \leq n-1$. Then*

$$\lim_{\nu \rightarrow \infty} \frac{\partial g_{\nu\alpha\bar{\beta}}}{\partial z_\gamma}(p_\nu) (\psi(p_\nu))^2$$

exists and is finite.

Proof. From (6.2), we obtain

$$\begin{aligned} \frac{\partial g_{\nu\alpha\bar{\beta}}}{\partial z_\gamma} \psi_\nu^2 &= \frac{\Lambda_{\nu\alpha\bar{\beta}\gamma} \psi_\nu^{2n}}{\Lambda_\nu \psi_\nu^{2n-2}} - \left(\frac{(\Lambda_{\nu\alpha\bar{\beta}} \psi_\nu^{2n-1})(\Lambda_{\nu\gamma} \psi_\nu^{2n-1})}{(\Lambda_\nu \psi_\nu^{2n-2})^2} + \frac{(\Lambda_{\nu\alpha\gamma} \psi_\nu^{2n})(\Lambda_{\nu\bar{\beta}} \psi_\nu^{2n-2})}{(\Lambda_\nu \psi_\nu^{2n-2})^2} + \frac{(\Lambda_{\nu\bar{\beta}\gamma} \psi_\nu^{2n-1})(\Lambda_{\nu\alpha} \psi_\nu^{2n-1})}{(\Lambda_\nu \psi_\nu^{2n-2})^2} \right. \\ &\quad \left. + \frac{2(\Lambda_{\nu\alpha} \psi_\nu^{2n-1})(\Lambda_{\nu\bar{\beta}} \psi_\nu^{2n-2})(\Lambda_{\nu\gamma} \psi_\nu^{2n-1})}{(\Lambda_\nu \psi_\nu^{2n-2})^3} \right). \end{aligned}$$

In view of theorem 1.2 and lemma 6.3 it is seen that the second and third terms have finite limits along $\{p_\nu\}$ and by lemma 6.5 the first term has finite limit along $\{p_\nu\}$. \square

Lemma 6.7. *The limit*

$$\lim_{\nu \rightarrow \infty} \det(g_{\nu\alpha\bar{\beta}}(p_\nu)) (\psi_\nu(p_\nu))^{n+1}$$

exists and is nonzero.

Proof. Let $(\Delta_{\alpha\bar{\beta}})$ be the cofactor matrix of $(g_{\alpha\bar{\beta}})$. Then expanding by the n -th row,

$$\det(g_{\alpha\bar{\beta}}) = g_{n\bar{1}} \Delta_{n\bar{1}} + \dots + g_{n\bar{n}} \Delta_{n\bar{n}}.$$

Therefore,

$$(6.23) \quad \det(g_{\alpha\bar{\beta}}) \psi^{n+1} = (g_{n\bar{1}} \psi^2)(\Delta_{n\bar{1}} \psi^{n-1}) + \dots + (g_{n\bar{n}} \psi^2)(\Delta_{n\bar{n}} \psi^{n-1}).$$

Note that

$$\begin{aligned} \Delta_{n\bar{\alpha}} \psi^{n-1} &= \psi^{n-1} (-1)^{n+\alpha} \det \begin{pmatrix} g_{1\bar{1}} & \dots & g_{1\bar{\alpha}-1} & g_{1\bar{\alpha}+1} & \dots & g_{1\bar{n}} \\ \dots & \dots & \dots & \dots & \dots & \dots \\ g_{n-1\bar{1}} & \dots & g_{n-1\bar{\alpha}-1} & g_{n-1\bar{\alpha}+1} & \dots & g_{n-1\bar{n}} \end{pmatrix} \\ &= (-1)^{n+\alpha} \det \begin{pmatrix} g_{1\bar{1}} \psi & \dots & g_{1\bar{\alpha}-1} \psi & g_{1\bar{\alpha}+1} \psi & \dots & g_{1\bar{n}} \psi \\ \dots & \dots & \dots & \dots & \dots & \dots \\ g_{n-1\bar{1}} \psi & \dots & g_{n-1\bar{\alpha}-1} \psi & g_{n-1\bar{\alpha}+1} \psi & \dots & g_{n-1\bar{n}} \psi \end{pmatrix} \end{aligned}$$

By lemma 6.4, if $1 \leq \alpha \leq n-1$ and $1 \leq \beta \leq n$, then the term $g_{\nu\alpha\bar{\beta}}(p_\nu) \psi_\nu(p_\nu)$ converges to a finite quantity. It follows that if $1 \leq \alpha \leq n-1$ then

$$\lim_{\nu \rightarrow \infty} \Delta_{\nu n\bar{\alpha}}(p_\nu) (\psi_\nu(p_\nu))^{n-1}$$

exists and is finite. Also if $1 \leq \alpha, \beta \leq n-1$, then $g_{\nu\alpha\bar{\beta}}(p_\nu) \psi_\nu(p_\nu)$ converges to $-(2n-2)\psi_{\alpha\bar{\beta}}(0)$. Therefore

$$\lim_{\nu \rightarrow \infty} \Delta_{\nu n\bar{n}}(p_\nu) (\psi_\nu(p_\nu))^{n-1} = (-1)^n (2n-2)^n \det(\psi_{\alpha\bar{\beta}}(0))_{1 \leq \alpha, \beta \leq n-1}.$$

Finally by lemma 6.1, if $1 \leq \alpha, \beta \leq n$, then $g_{\nu\alpha\bar{\beta}}(p_\nu) (\psi_\nu(p_\nu))^2$ converges to $(2n-2)\psi_\alpha(0)\psi_{\bar{\beta}}(0)$. Now it follows from (6.23) that

$$\lim_{\nu \rightarrow \infty} \det(g_{\nu\alpha\bar{\beta}}(p_\nu)) (\psi_\nu(p_\nu))^{n+1} = (-1)^n (2n-2)^{n+1} \det(\psi_{\alpha\bar{\beta}}(0))_{1 \leq \alpha, \beta \leq n-1} \neq 0$$

as D is strongly pseudoconvex at 0. \square

Proof of Theorem 1.1: We have

$$-\frac{1}{(g_{n\bar{n}}(z))^2} \frac{\partial^2 g_{n\bar{n}}}{\partial z_n \partial \bar{z}_n}(z) = -\frac{1}{(g_{n\bar{n}}(z)(\psi(z))^2)^2} \frac{\partial^2 g_{n\bar{n}}}{\partial z_n \partial \bar{z}_n}(z) (\psi(z))^4$$

By lemma 6.1

$$-\frac{1}{(g_{\nu n\bar{n}}(p_\nu))^2} \frac{\partial^2 g_{\nu n\bar{n}}}{\partial z_n \partial \bar{z}_n}(p_\nu) \rightarrow -\frac{1}{\{(2n-2)\psi_n(0)\psi_{\bar{n}}(0)\}^2} \{6(2n-2)\psi_n(0)\psi_{\bar{n}}(0)\psi_n(0)\psi_{\bar{n}}(0)\} = -\frac{3}{n-1}.$$

To compute the limit of the second term note that $g^{\beta\bar{\alpha}} = \Delta_{\alpha\bar{\beta}} / \det(g_{\alpha\bar{\beta}})$. There are various cases to be considered depending on α and β .

Case 1: $\alpha \neq n, \beta \neq n$. Here

$$\frac{1}{g_{n\bar{\alpha}}^2} g^{\beta\bar{\alpha}} \frac{\partial g_{n\bar{\alpha}}}{\partial z_n} \frac{\partial g_{\beta\bar{n}}}{\partial \bar{z}_n} = \frac{1}{(g_{n\bar{n}}\psi^2)^2 (\det(g_{i\bar{j}})\psi^{n+1})} (\Delta_{\alpha\bar{\beta}}\psi^n) \left(\frac{\partial g_{n\bar{\alpha}}}{\partial z_n} \psi^2 \right) \left(\frac{\partial g_{\beta\bar{n}}}{\partial \bar{z}_n} \psi^3 \right)$$

By lemma 6.1,

$$g_{\nu n\bar{n}}(p_\nu) (\psi_\nu(p_\nu))^2 \rightarrow (2n-2)$$

By lemma 6.7, $\det(g_{i\bar{j}}(p_\nu)) (\psi_\nu(p_\nu))^{n+1}$ converges to a nonzero finite. Also

$$\Delta_{\alpha\bar{\beta}} = \sum_{\sigma} (-1)^{sgn(\sigma)} g_{1\sigma(1)} \overline{g_{2\sigma(2)}} \cdots \overline{g_{n\sigma(n)}}$$

where the summation runs over all permutations

$$\sigma : \{1, \dots, \alpha-1, \alpha+1, \dots, n\} \rightarrow \{1, \dots, \beta-1, \beta+1, \dots, n\}$$

Hence

$$\Delta_{\alpha\bar{\beta}}\psi^n = \sum_{\sigma} (-1)^{sgn(\sigma)} (g_{1\sigma(1)}\psi)(g_{2\sigma(2)}\psi) \cdots (g_{n\sigma(n)}\psi^2).$$

By lemma 6.4, for $1 \leq i \leq n-1$, $g_{\nu i\sigma(i)}(p_\nu) (\psi_\nu(p_\nu))$ converges to a finite quantity. Also

$$g_{\nu n\sigma(n)}(p_\nu) (\psi(p_\nu))^2 \rightarrow (2n-2)\psi_n(0)\overline{\psi_{\sigma(n)}(0)}$$

by lemma 6.1. Thus $\Delta_{\nu\alpha\bar{\beta}}(p_\nu) (\psi_\nu(p_\nu))^n$ converges to a finite quantity.

By lemma 6.6, $\frac{\partial g_{\nu n\bar{\alpha}}}{\partial z_n}(p_\nu) (\psi_\nu(p_\nu))^2$ converges to a finite quantity and by lemma 6.1

$$\frac{\partial g_{\nu\beta\bar{n}}}{\partial \bar{z}_n}(p_\nu) (\psi_\nu(p_\nu))^3 = \overline{\left(\frac{\partial g_{\nu n\bar{\beta}}}{\partial z_n}(p_\nu) (\psi_\nu(p_\nu))^3 \right)} \rightarrow -2(2n-2) \overline{(\psi_{\nu n}(0))} \overline{(\psi_{\bar{\beta}}(0))} \overline{(\psi_n(0))} = 0.$$

Hence

$$\lim_{\nu \rightarrow \infty} \frac{1}{(g_{\nu n\bar{n}}(p_\nu))^2} g_{\nu}^{\beta\bar{\alpha}}(p_\nu) \frac{\partial g_{\nu n\bar{\alpha}}}{\partial z_n}(p_\nu) \frac{\partial g_{\nu\beta\bar{n}}}{\partial \bar{z}_n}(p_\nu) = 0.$$

Case 2: $\alpha = n, \beta \neq n$. Here

$$\frac{1}{g_{n\bar{n}}^2} g^{\beta\bar{n}} \frac{\partial g_{n\bar{n}}}{\partial z_n} \frac{\partial g_{\beta\bar{n}}}{\partial \bar{z}_n} = \frac{1}{(g_{n\bar{n}}\psi^2)^2 (\det(g_{i\bar{j}})\psi^{n+1})} (\Delta_{n\bar{\beta}}\psi^{n-1}) \left(\frac{\partial g_{n\bar{n}}}{\partial z_n} \psi^3 \right) \left(\frac{\partial g_{\beta\bar{n}}}{\partial \bar{z}_n} \psi^3 \right)$$

By lemma 6.1,

$$g_{\nu n\bar{n}}(p_\nu) (\psi_\nu(p_\nu))^2 \rightarrow (2n-2).$$

By lemma 6.7, $\det(g_{\nu\alpha\bar{\beta}}(p_\nu)) (\psi(p_\nu))^{n+1}$ has a nonzero limit and $\Delta_{\nu n\bar{\beta}}(p_\nu) (\psi_\nu(p_\nu))^{n-1}$ converges to a finite quantity. By lemma 6.1,

$$\frac{\partial g_{\nu n\bar{n}}}{\partial z_n}(p_\nu) (\psi_\nu(p_\nu))^3 \rightarrow -2(2n-2)\psi_n(0)\overline{\psi_{\bar{n}}(0)}\psi_n(0) = -2(2n-2)$$

and

$$\frac{\partial g_{\nu\beta\bar{n}}}{\partial \bar{z}_n}(p_\nu) (\psi_\nu(p_\nu))^3 = \overline{\left(\frac{\partial g_{\nu n\bar{\beta}}}{\partial z_n}(p_\nu) (\psi_\nu(p_\nu))^3 \right)} \rightarrow -2(2n-2) \overline{(\psi_n(0))} \overline{(\psi_{\bar{\beta}}(0))} \overline{(\psi_n(0))} = 0.$$

Hence

$$\lim_{\nu \rightarrow \infty} \frac{1}{(g_{\nu n\bar{n}}(p_\nu))^2} g_{\nu}^{\beta\bar{n}}(p_\nu) \frac{\partial g_{\nu n\bar{n}}}{\partial z_n}(p_\nu) \frac{\partial g_{\nu\beta\bar{n}}}{\partial \bar{z}_n}(p_\nu) = 0$$

Case 3: $\alpha \neq n$ and $\beta = n$. This case is similar to Case 2 and we have

$$\lim_{\nu \rightarrow \infty} \frac{1}{(g_{\nu n\bar{n}}(p_\nu))^2} g_{\nu}^{n\bar{\alpha}}(p_\nu) \frac{\partial g_{\nu n\bar{\alpha}}}{\partial z_n}(p_\nu) \frac{\partial g_{\nu n\bar{n}}}{\partial \bar{z}_n}(p_\nu) = 0.$$

Case 4: $\alpha = n, \beta = n$. In this case we have

$$\frac{1}{g_{n\bar{n}}^2} g^{n\bar{n}} \frac{\partial g_{n\bar{n}}}{\partial z_n} \frac{\partial g_{n\bar{n}}}{\partial \bar{z}_n} = \frac{1}{(g_{n\bar{n}}\psi^2)^2 (\det(g_{i\bar{j}})\psi^{n+1})} (\Delta_{n\bar{n}}\psi^{n-1}) \left(\frac{\partial g_{n\bar{n}}}{\partial z_n} \psi^3 \right) \left(\frac{\partial g_{n\bar{n}}}{\partial \bar{z}_n} \psi^3 \right).$$

From lemma 6.1,

$$g_{\nu n\bar{n}}(p_\nu) (\psi_\nu(p_\nu))^2 \rightarrow (2n-2)$$

and both

$$\frac{\partial g_{\nu n \bar{n}}}{\partial z_n}(p_\nu)(\psi_\nu(p_\nu))^3, \frac{\partial g_{n \bar{n}}}{\partial \bar{z}_n}(p_\nu)(\psi_\nu(p_\nu))^3 \rightarrow -2(2n-2).$$

From lemma 6.7

$$\Delta_{\nu n \bar{n}}(p_\nu)(\psi_\nu(p_\nu))^{n-1} \rightarrow (-1)^n(2n-2)^n \det(\psi_{i\bar{j}}(0))_{1 \leq i, j \leq n-1}$$

and

$$\det(g_{\nu i \bar{j}}(p_\nu))(\psi_\nu(p_\nu))^{n+1} \rightarrow (-1)^n(2n-2)^{n+1} \det(\psi_{i\bar{j}}(0))_{1 \leq i, j \leq n-1}.$$

Hence

$$\lim_{\nu \rightarrow \infty} \frac{1}{(g_{\nu n \bar{n}}(p_\nu))^2} g_{\nu n \bar{n}}(p_\nu) \frac{\partial g_{\nu n \bar{n}}}{\partial z_n}(p_\nu) \frac{\partial g_{\nu n \bar{n}}}{\partial \bar{z}_n}(p_\nu) = \frac{2}{n-1}.$$

From the various cases we finally obtain

$$\lim_{\nu \rightarrow \infty} R(z_\nu, v_N(z_\nu)) = -3/(n-1) + 2/(n-1) = -1/(n-1).$$

7. EXISTENCE OF CLOSED GEODESICS

In this section we prove theorem 1.4. The main tool that we will use is the following theorem of Herbort [6]:

Theorem 7.1. *Let G be a bounded domain in \mathbf{R}^k , such that $\pi_1(G)$ is nontrivial. Assume that the following conditions are satisfied:*

- (i) *For each $p \in \bar{G}$ there is an open neighbourhood $U \subset \mathbf{R}^k$, such that the set $G \cap U$ is simply connected.*
- (ii) *G is equipped with a complete Riemannian metric g which possesses the following property:*
 (P) *For each $S > 0$ there is a $\delta > 0$, such that for every point $p \in G$ with $d(p, \partial D) < \delta$ and every $X \in \mathbf{R}^k$, $g(p, X) \geq S|X|^2$.*

Then every nontrivial homotopy class in $\pi_1(G)$ contains a closed geodesic for g .

In [1] we proved the following boundary behaviour of the Λ -metric: Let D be a C^∞ -smoothly bounded strongly pseudoconvex domain in \mathbf{C}^n and ds^2 be the Λ -metric on D . Suppose that ψ is any C^∞ -smooth defining function for D . Then

$$ds_z^2(v, v) \approx \delta^{-2}(z)|v_N(z)|^2 + \delta^{-1}(z)\mathcal{L}_\psi(\pi(z), v_H(z))$$

uniformly for all z sufficiently close to ∂D and all $v \in \mathbf{C}^n$. Here, $v = v_H(z) + v_N(z)$ is as usual the decomposition of v at the point $\pi(z) \in \partial D$, $\delta(z) = d(z, \partial D)$ is the Euclidean distance of z to the boundary of D and $\mathcal{L}_\psi(z, v)$ denotes the Levi form of ψ at z along v , i.e.,

$$\mathcal{L}_\psi(z, v) = \sum_{\alpha, \beta=1}^n \frac{\partial^2 \psi}{\partial z_\alpha \partial \bar{z}_\beta}(z) v^\alpha \bar{v}^\beta.$$

Also, it is known that the Bergman metric ds_B^2 on D has the same boundary behaviour. It follows that

$$ds_z^2(v, v) \approx ds_{Bz}^2(v, v)$$

uniformly for all z sufficiently close to ∂D and all $v \in \mathbf{C}^n$. Also, on compact subsets of D , these two metrics are uniformly comparable to the Euclidean metric. Thus we have the following:

Proposition 7.2. *Let D be a C^∞ -smoothly bounded strongly pseudoconvex domain in \mathbf{C}^n . Let ds^2 denotes the Λ -metric on D and ds_B^2 denotes the Bergman metric on D . Then there exists a constant $C > 1$ such that*

$$C^{-1}ds_B^2 \leq ds^2 \leq Cds_B^2$$

uniformly on D .

Proof of theorem 1.4. We will show that the Λ -metric on D satisfies the hypothesis of theorem 7.1. Indeed, since ∂D is smooth, condition (i) is evidently satisfied. Also, note that the Bergman metric is complete on D ([8]) and satisfies property (P) [2]. It follows from proposition 7.2 that condition (ii) is satisfied. Thus the theorem is proved.

8. L^2 -COHOMOLOGY OF THE Λ -METRIC

Let M be a complete Kähler manifold of complex dimension n . Let Ω_2^i be the space of square integrable i -forms on M . Then the (reduced) L^2 -cohomology of the complex

$$\Omega_2^0(M) \xrightarrow{d_0} \Omega_2^1(M) \xrightarrow{d_1} \dots \xrightarrow{d_{2n-1}} \Omega_2^{2n}(M) \xrightarrow{d_{2n}} 0$$

is defined by

$$H_2^i(M) = \frac{\ker d_i}{\overline{\operatorname{Im} d_{i-1}}}$$

where the closure is taken in L^2 . Now, let $\mathcal{H}_2^i(M)$ be the space of square integrable harmonic i -forms on M . Then the completeness of the metric implies that $H_2^i(M) \cong \mathcal{H}_2^i(M)$. We have the following result ([3]) on the vanishing of the L^2 -cohomology outside the middle dimension:

Proposition 8.1. *Let M be a complete Kähler manifold of complex dimension n . Suppose that the Kähler form ω of M can be written as $\omega = d\eta$, where η is bounded in supremum norm. Then $\mathcal{H}_2^i(M) = 0$ for $i \neq n$.*

Also, we have the following result ([9]) on the infinite dimensionality of the L^2 -cohomology of the middle dimension:

Theorem 8.2. *Let D be a domain in a connected complex manifold of dimension n and ds^2 be a Hermitian metric on D . Suppose that there exists a non-degenerate regular boundary point $z_0 \in \partial D$. Also, suppose that there exist a neighbourhood U of z_0 , a local defining function ϕ for D defined on U and a Hermitian metric ds_U^2 defined on U such that*

$$C^{-1}ds^2 < (-\phi)^{-a}ds_U^2 + (-\phi)^{-b}\partial\phi\bar{\partial}\phi < Cds^2$$

on $U \cap D$, where a, b and C are positive numbers with $1 \leq a \leq b < a + 3$. Then, for any positive integer p and q with $p + q = n$,

$$\dim H_2^{p,q}(D) = \infty$$

where $H_2^{p,q}(D)$ denotes the L^2 $\bar{\partial}$ -cohomology group relative to ds^2 .

Remark 8.3. The above theorem in particular implies that if ds^2 is complete and Kähler, then for any positive integer p and q with $p + q = n$,

$$\dim \mathcal{H}_2^{p,q}(D) = \infty$$

where $\mathcal{H}_2^{p,q}(D)$ is the space of square integrable harmonic (p, q) -forms on D relative to ds^2 .

To apply these results to the Λ metric, let D be a C^∞ -smoothly bounded pseudoconvex domain in \mathbb{C}^n and ds^2 be the Λ -metric on D . Then the Kähler form ω of ds^2 is given by

$$\omega = i \sum_{\alpha=1}^n \frac{\partial^2 \log(-\Lambda)}{\partial z_\alpha \partial \bar{z}_\beta} dz_\alpha \wedge d\bar{z}_\beta = d\eta$$

where

$$\eta = -i \sum_{\alpha=1}^n \frac{\partial \log(-\Lambda)}{\partial z_\alpha} dz_\alpha.$$

Now let ψ be a C^∞ -smooth defining function for D . Then, differentiating the relation

$$\lambda = \Lambda\psi^{2n-2}$$

with respect to z_α we obtain

$$(8.1) \quad \frac{\partial \log(-\Lambda)}{\partial z_\alpha} = \lambda^{-1} \lambda_\alpha - 2(n-1)\psi^{-1}\psi_\alpha.$$

Therefore,

$$(8.2) \quad \eta(v) = -i \sum_{\alpha=1}^n \frac{\partial \log(-\Lambda)}{\partial z_\alpha} v^\alpha = -i(\lambda^{-1}\langle v, \bar{\partial}\lambda \rangle - 2(n-1)\psi^{-1}\langle v, \bar{\partial}\psi \rangle)$$

and

$$(8.3) \quad |\eta(v)|^2 = \lambda^{-2}|\langle v, \bar{\partial}\lambda \rangle|^2 - 4(n-1)\lambda^{-1}\psi^{-1}\Re(\langle v, \bar{\partial}\lambda \rangle \overline{\langle v, \bar{\partial}\psi \rangle}) + 4(n-1)^2\psi^{-2}|\langle v, \bar{\partial}\psi \rangle|^2$$

Also, differentiating (8.1) with respect to \bar{z}_β we obtain

$$(8.4) \quad \frac{\partial^2 \log(-\Lambda)}{\partial z_\alpha \partial \bar{z}_\beta} = \lambda^{-1} \lambda_{\alpha\bar{\beta}} - \lambda^{-2} \lambda_\alpha \lambda_{\bar{\beta}} + 2(n-1)\psi^{-2}\psi_\alpha \psi_{\bar{\beta}} - 2(n-1)\psi^{-1}\psi_{\alpha\bar{\beta}}.$$

Therefore,

$$(8.5) \quad ds^2(v, v) = \sum_{\alpha, \beta=1}^n \frac{\partial^2 \log(-\Lambda)}{\partial z_\alpha \partial \bar{z}_\beta} v^\alpha \bar{v}^\beta \\ = \lambda^{-1} \mathcal{L}_\lambda(z, v) - \lambda^{-2} |\langle v, \bar{\partial} \lambda \rangle|^2 + 2(n-1) \psi^{-2} |\langle v, \bar{\partial} \psi \rangle|^2 - 2(n-1) \psi^{-1} \mathcal{L}_\psi(z, v)$$

Lemma 8.4. *Let D be a C^∞ -smoothly bounded pseudoconvex domain in \mathbf{C}^n and ψ be a C^∞ smooth defining function for D . One has the following:*

(1) *If $z_0 \in \partial D$ and $v \in \mathbf{C}^n$ is a unit vector satisfying $\langle v, \bar{\partial} \bar{\psi}(z_0) \rangle \neq 0$, then*

$$\lim_{z \rightarrow z_0} \frac{|\eta_z(v)|^2}{ds_z^2(v, v)} = 2(n-1),$$

(2) *If $z_0 \in \partial D$ is strongly pseudoconvex and $v \in \mathbf{C}^n$ is a unit vector satisfying $\langle v, \bar{\partial} \bar{\psi}(z_0) \rangle = 0$, then*

$$\lim_{z \rightarrow z_0} \frac{|\eta_z(v)|^2}{ds_z^2(v, v)} = 0.$$

Moreover, the limits are approached uniformly for $z_0 \in \partial D$ and unit vectors v .

Proof. Since λ is C^2 -smooth up to \bar{D} and ψ is C^∞ -smooth, the terms

$$\langle v, \bar{\partial} \bar{\lambda}(z) \rangle, \quad \langle v, \bar{\partial} \bar{\psi}(z) \rangle, \quad \mathcal{L}_\lambda(z, v), \quad \text{and} \quad \mathcal{L}_\psi(z, v)$$

are uniformly bounded for all $z \in \bar{D}$ and all $v \in \mathbf{C}^n$ with $|v| = 1$. Also, since $\lambda = -|\partial \psi|^{2n-2}$ on ∂D , it is evident that λ^{-1} is bounded near ∂D .

By the above observation it is evident from (8.3) that

$$\lim_{z \rightarrow z_0} (\psi(z))^2 |\eta_z(v)|^2 = 4(n-1)^2 |\langle v, \bar{\partial} \psi(z_0) \rangle|^2$$

and from (8.5) that

$$\lim_{z \rightarrow z_0} (\psi(z))^2 ds_z^2(v, v) = 2(n-1) |\langle v, \bar{\partial} \psi(z_0) \rangle|^2$$

uniformly for $z_0 \in \partial D$ and unit vector v . Therefore,

$$\lim_{z \rightarrow z_0} \frac{|\eta_z(v)|^2}{ds_z^2(v, v)} = 2(n-1)$$

uniformly for $z_0 \in \partial D$ and unit vector v satisfying $\langle v, \bar{\partial} \psi(z_0) \rangle \neq 0$, which proves (1).

To prove (2), observe that if $\langle v, \bar{\partial} \psi(z_0) \rangle = 0$ then

$$\langle v, \bar{\partial} \psi(z) \rangle = \langle v, \bar{\partial} \psi(z) \rangle - \langle v, \bar{\partial} \psi(z_0) \rangle = \langle v, \bar{\partial} \psi(z) - \bar{\partial} \psi(z_0) \rangle.$$

Since

$$|\bar{\partial} \psi(z) - \bar{\partial} \psi(z_0)| \lesssim (-\psi(z))$$

uniformly for z near z_0 , it follows that

$$|\langle v, \bar{\partial} \psi(z) \rangle| \lesssim (-\psi(z))$$

uniformly for z near z_0 and unit vectors v satisfying $\langle v, \bar{\partial} \psi(z_0) \rangle = 0$. Combining this with our previous observation, it now follows from (8.3) that

$$\lim_{z \rightarrow z_0} (-\psi(z)) |\eta_z(v)|^2 = 0$$

and from (8.5) that

$$\lim_{z \rightarrow z_0} (-\psi(z)) ds_z^2(v, v) = 2(n-1) \mathcal{L}_\psi(z_0, v)$$

uniformly for $z_0 \in \partial D$ and unit vectors v satisfying $\langle v, \bar{\partial} \psi(z_0) \rangle = 0$. Since z_0 is a strongly pseudoconvex boundary point, $\mathcal{L}_\psi(z_0, v) > 0$ and hence

$$\lim_{z \rightarrow z_0} \frac{|\eta_z(v)|^2}{ds_z^2(v, v)} = 0$$

uniformly for all strongly pseudoconvex boundary points $z_0 \in \partial D$ and all unit vectors v satisfying $\langle v, \bar{\partial} \psi(z_0) \rangle = 0$, which proves (2). \square

Proposition 8.5. *Let D be a C^∞ -smoothly bounded strongly pseudoconvex domain in \mathbf{C}^n . Then the ratio*

$$(8.6) \quad \frac{|\eta_z(v)|^2}{ds_z^2(v, v)}$$

is uniformly bounded for $z \in D$ and vectors $v \in \mathbf{C}^n$ with $v \neq 0$.

Proof. By lemma 8.4, the ratio

$$\frac{|\eta_z(v)|^2}{ds_z^2(v, v)}$$

is uniformly bounded for all z near ∂D and all unit vectors v . It is evident that this ratio is uniformly bounded for all z on a compact subset of D and all unit vectors v . Now, by homogeneity of $\eta_z(v)$ and $ds_z^2(v, v)$ in the vector variable v it follows that the ratio is uniformly bounded above for all $z \in D$ and vectors $v \neq 0$. \square

We also note the following:

Proposition 8.6. *Let D be a C^∞ -smoothly bounded strongly pseudoconvex domain in \mathbf{C}^n and ds^2 be the Λ -metric on D . Suppose that ψ is a C^∞ -smooth defining function for D . Then*

$$ds^2 \approx (-\psi)^{-1} ds_E^2 + (-\psi)^{-2} \partial\psi\bar{\partial}\psi$$

uniformly near ∂D , where ds_E^2 is the Euclidean metric on \mathbf{C}^n .

Proof. It is known that the Bergman metric on D satisfies the same estimate. Therefore, the proof follows from proposition 7.2. \square

Proof of theorem 1.5. Let ds^2 be the Λ -metric on D . By proposition 7.2 and the completeness of the Bergman metric on D , ds^2 is complete. Therefore, by propositions 8.1 and 8.5, we have

$$\mathcal{H}_2^i(D) = 0$$

for $i \neq n$ and hence

$$\mathcal{H}_2^{p,q}(D) = 0$$

for $p + q \neq n$. Also, by remark 8.3 and proposition 8.6,

$$\dim \mathcal{H}_2^{p,q}(D) = \infty$$

for any positive integers p and q with $p + q = n$. Moreover, a harmonic $(n, 0)$ form on D is precisely of the form

$$f(z) dz_1 \wedge \dots \wedge dz_n$$

where $f(z)$ is a harmonic function (with respect to the standard Laplacian) on D . Therefore, $\mathcal{H}_2^{n,0}(D)$ (and thus $\mathcal{H}_2^{0,n}(D)$) is isomorphic to the space of square integrable harmonic functions on D which is evidently infinite dimensional. This completes the proof. \square

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